Germán Sierra

Instituto de Física Teórica UAM-CSIC, Madrid, 22 February 2017

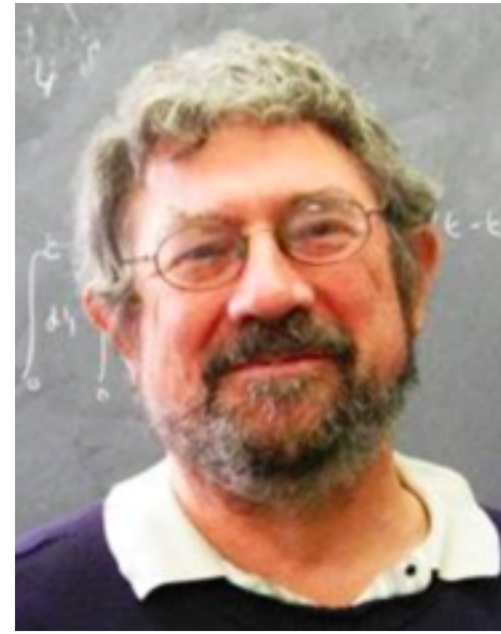
The winners

David J. Thouless
1934 – Scotland
Whashington Univ.

1/2 Prize

F. Duncan M. Haldane
(1951 – London)
Princeton Univ.

1/4 Prize



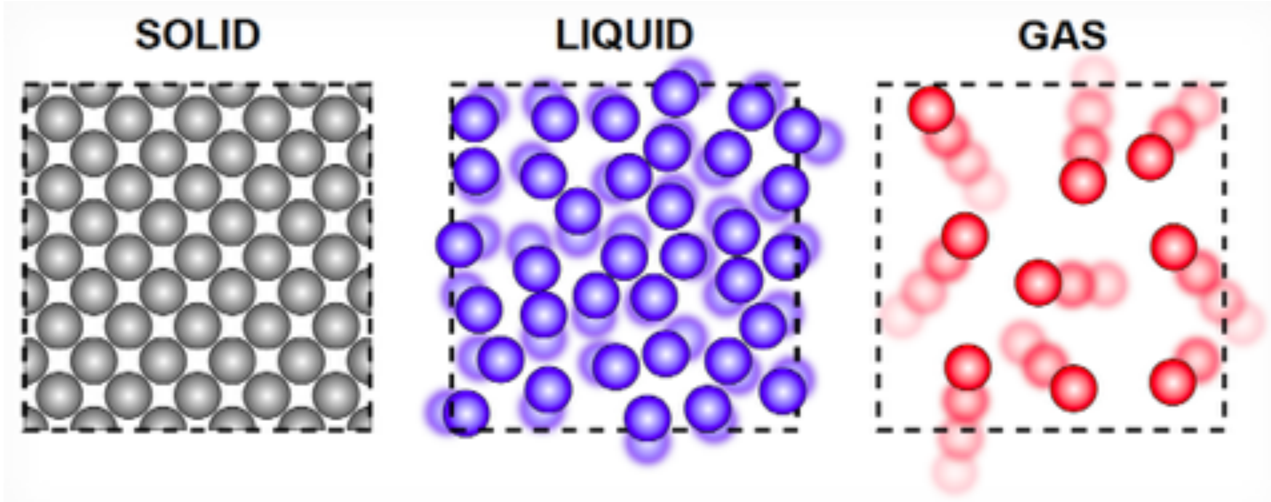
J. Michael Kosterlitz
1943 – Aberdeen (UK)
Brown Univ.

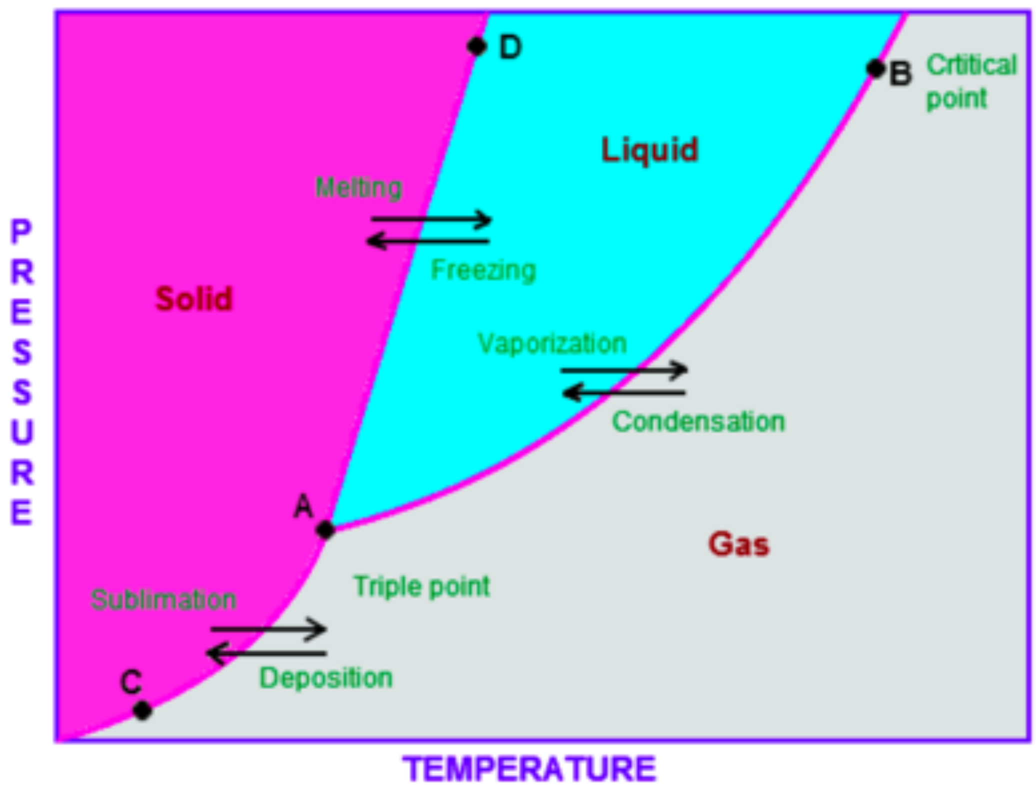
1/4 Prize

Prize Motivation:

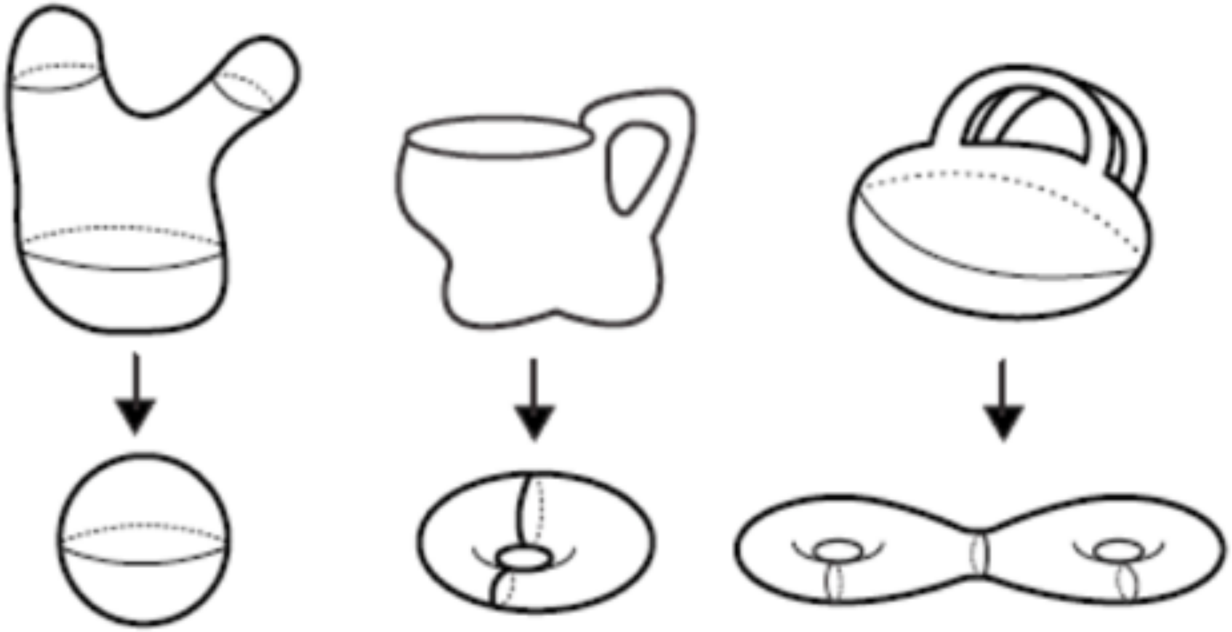
"for theoretical discoveries
of
topological phase transitions
and
topological phases of matter"

First time the term “topological” appears in a Nobel prize in Physics

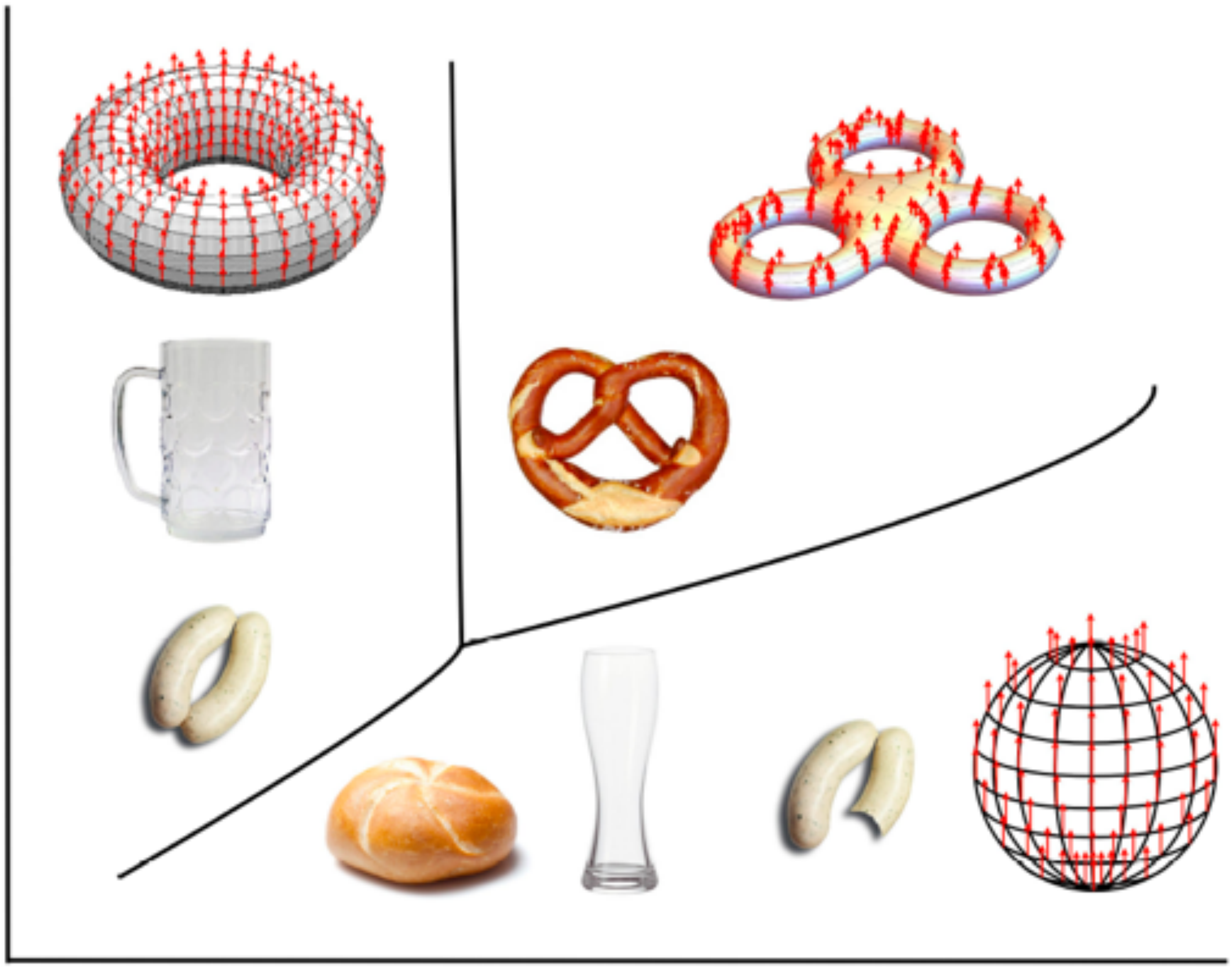

SOLID
LIQUID
GAS


D
B
Crtitical point
Liquid
Melting
Freezing
Solid
Vaporization
Condensation
A
Gas
Triple point
Sublimation
Deposition
C
PRESSURE
TEMPERATURE

Topology



Cartoon of a Topological Phase Diagram



Nathan Goldman

A historical note

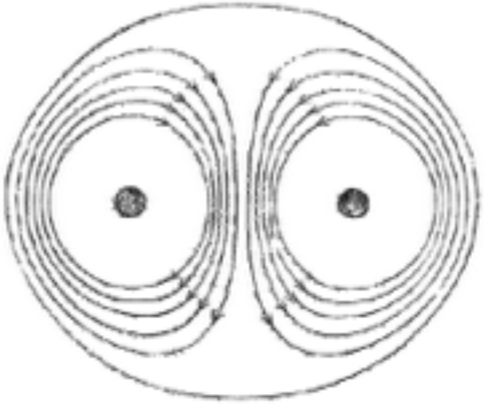
1824- 1907

On Vortex Atoms

By Lord Kelvin (Sir William Thomson)

Proceedings of the Royal Society of Edinburgh, Vol. VI, 1867, pp. 94-105.

After noticing Helmholtz's admirable discovery of the law of vortex motion in a perfect liquid- that is, in a fluids perfectly destitute of viscosity (or fluid friction)--the author said that this discovery inevitable suggests the idea that Helmholtz's rings are the only true atoms.



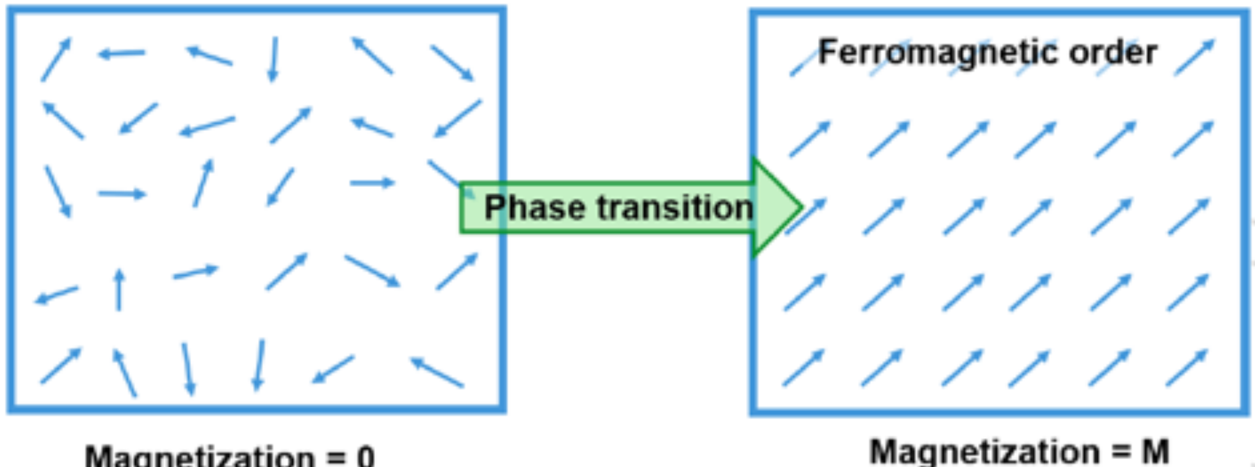
Standard Model of Phase transitions

Order Parameter = average of a Local Observable $\langle O(\vec{r}) \rangle$

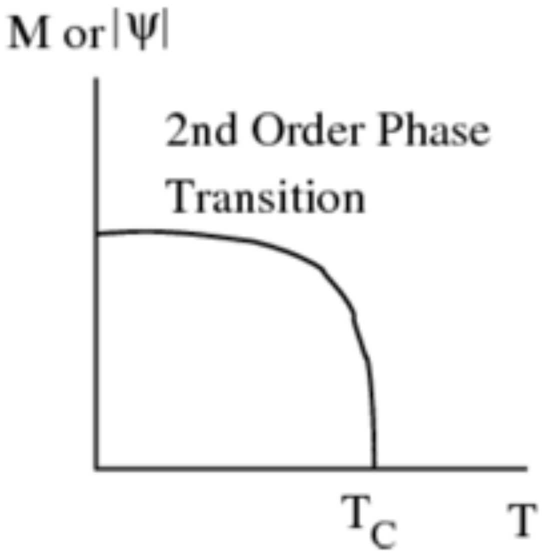
Ordered phase $\langle O(\vec{r}) \rangle \neq 0$ Low Temperature $T < T_c$

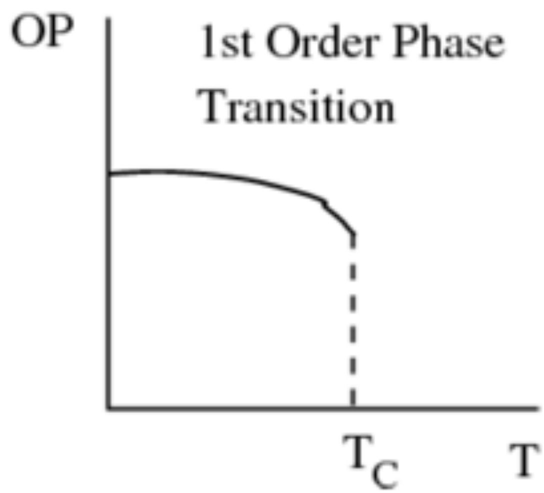
Disordered phase $\langle O(\vec{r}) \rangle = 0$ High Temperature $T > T_c$

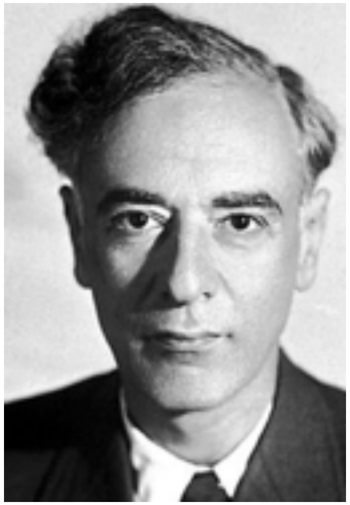
Example: magnetization/spin $\vec{M}(\vec{r}) = \sum_i \langle \vec{S}_i \rangle$



Phase transitions

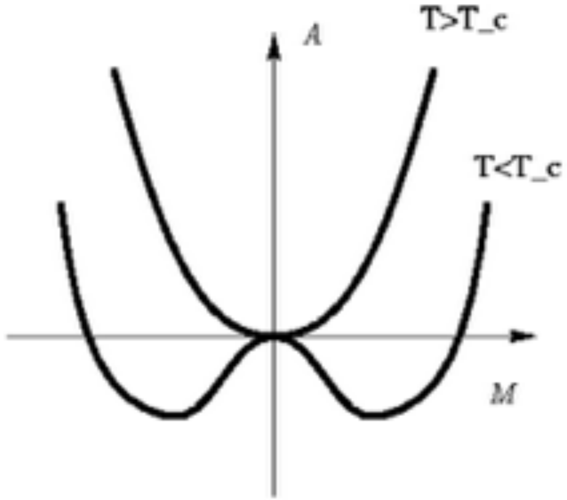




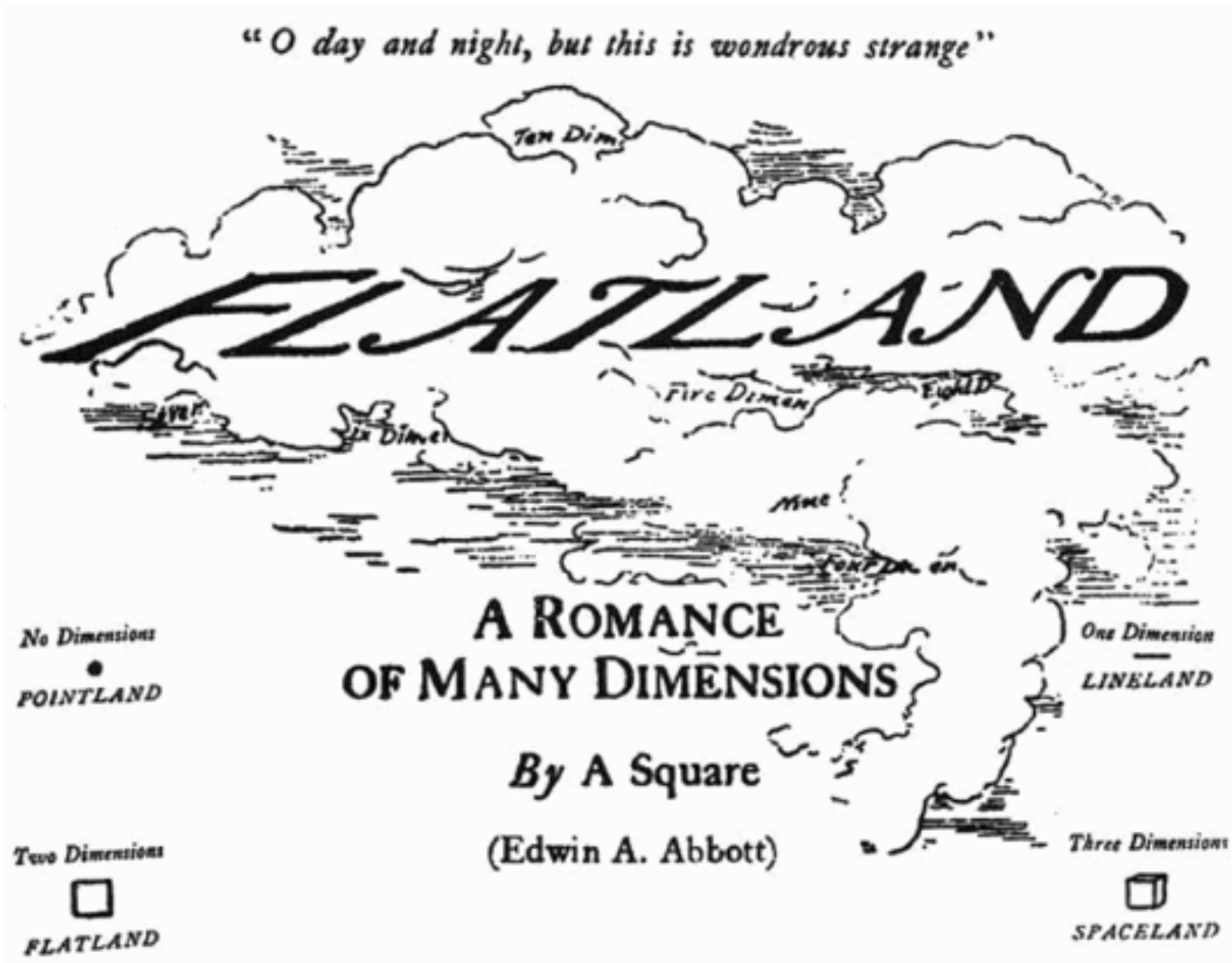


Lev Landau
Nobel 1962

Spontaneous Symmetry Breaking



$\frac{1}{4}+\frac{1}{4}$ Prize


"O day and night, but this is wondrous strange"

FLATLAND

A ROMANCE OF MANY DIMENSIONS

By A Square

(Edwin A. Abbott)

No Dimensions

POINTLAND

One Dimension

LINELAND

Two Dimensions

FLATLAND

Three Dimensions

SPACELAND

Phase transitions in 2 dimensions

Rudolf Peierls
1907-1995

Peierls argued in 1935 that thermal motions
of long wave phonons distroy long range order in 2D:

$$\left\langle (\vec{r}_i - \vec{r}_{i,0})^2 \right\rangle \propto \log L$$

Mermin-Wagner theorem (1966):

A continuous symmetry cannot be broken spontaneously at finite T in dimensions

$$d \leq 2$$

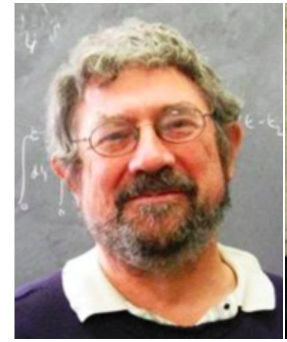
David Mermin

Herbert Wagner

Reason: Goldstone bosons produced infrared divergences in correlations

A discrete symmetry can be broken spontaneously. Example: the Ising model $s_i \leftrightarrow -s_i$

Ordering, metastability and phase transitions in two-dimensional systems




J M Kosterlitz and D J Thouless
Department of Mathematical Physics, University of Birmingham, Birmingham B15 2TT, UK

Received 13 November 1972

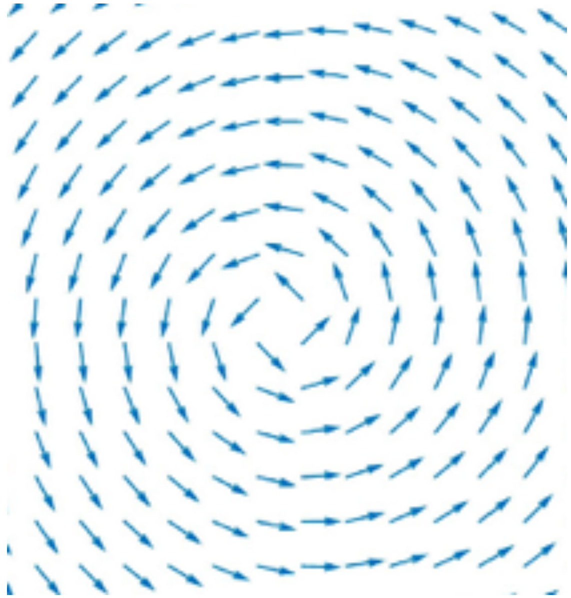
Abstract. A new definition of order called topological order is proposed for two-dimensional systems in which no long-range order of the conventional type exists. The possibility of a phase transition characterized by a change in the response of the system to an external perturbation is discussed in the context of a mean field type of approximation. The critical behaviour found in this model displays very weak singularities. The application of these ideas to the *xy* model of magnetism, the solid–liquid transition, and the neutral superfluid are discussed. This type of phase transition cannot occur in a superconductor nor in a Heisenberg ferromagnet, for reasons that are given.

J. Phys. C: Solid State Phys., Vol. 6, 1973. Printed in Great Britain.

Topological excitations

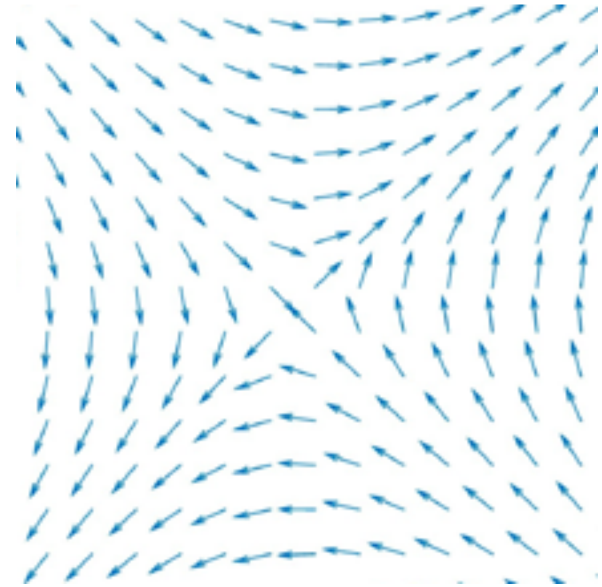
vortex

$$\Delta\theta = 2\pi$$



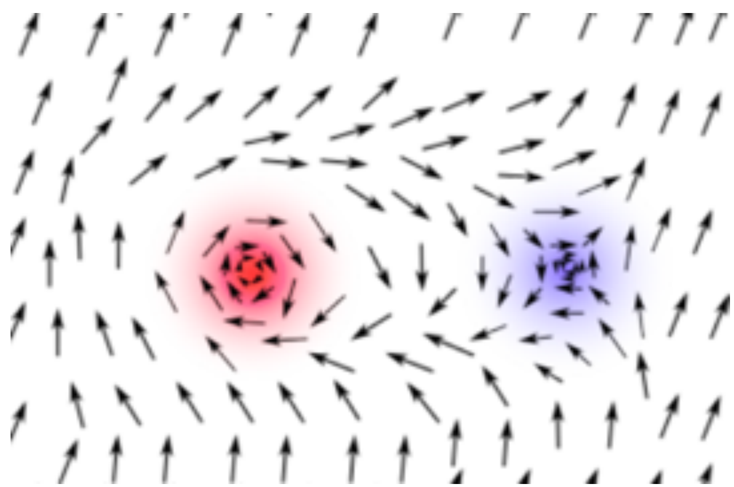
antivortex

$$\Delta\theta = -2\pi$$



Vortex-antivortex pair

$$\Delta\theta = 2\pi - 2\pi = 0$$



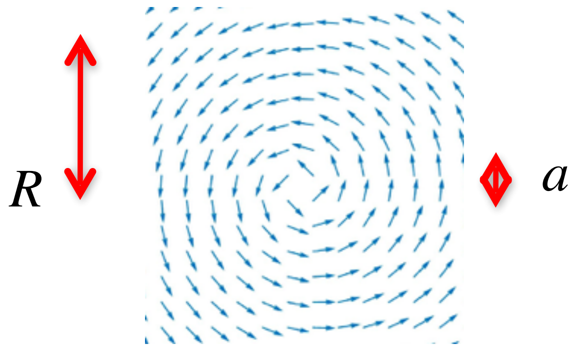
(lord Kelvin !!)

An elegant thermodynamic argument

Classical spin in 2D $\quad \vec{S}_i = (\cos\theta_i, \sin\theta_i)$

Hamiltonian of XY model $\quad H_{XY} = -J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j = -J \sum_{\langle i,j \rangle} \cos\left(\theta_i - \theta_j\right) \approx \frac{J}{2} \int d^2\vec{r}\, (\nabla\theta)^2$

A vortex configuration



Energy / Entropy

$$E_v = \pi J \log(R/a)$$

$$S_v = k_B \log(R^2/a^2)$$

Free Energy $\quad F = E_v - T S_v = (\pi J - 2k_B T)\ \log(R/a)$

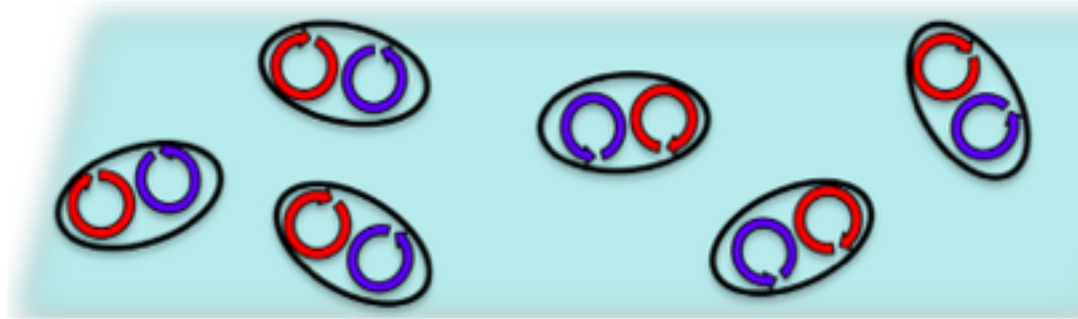
Low temperatures -> energy dominates -> $F > 0$ *no free vortex*

Hight temperatures -> entropy dominates -> $F < 0$ *free vortex*

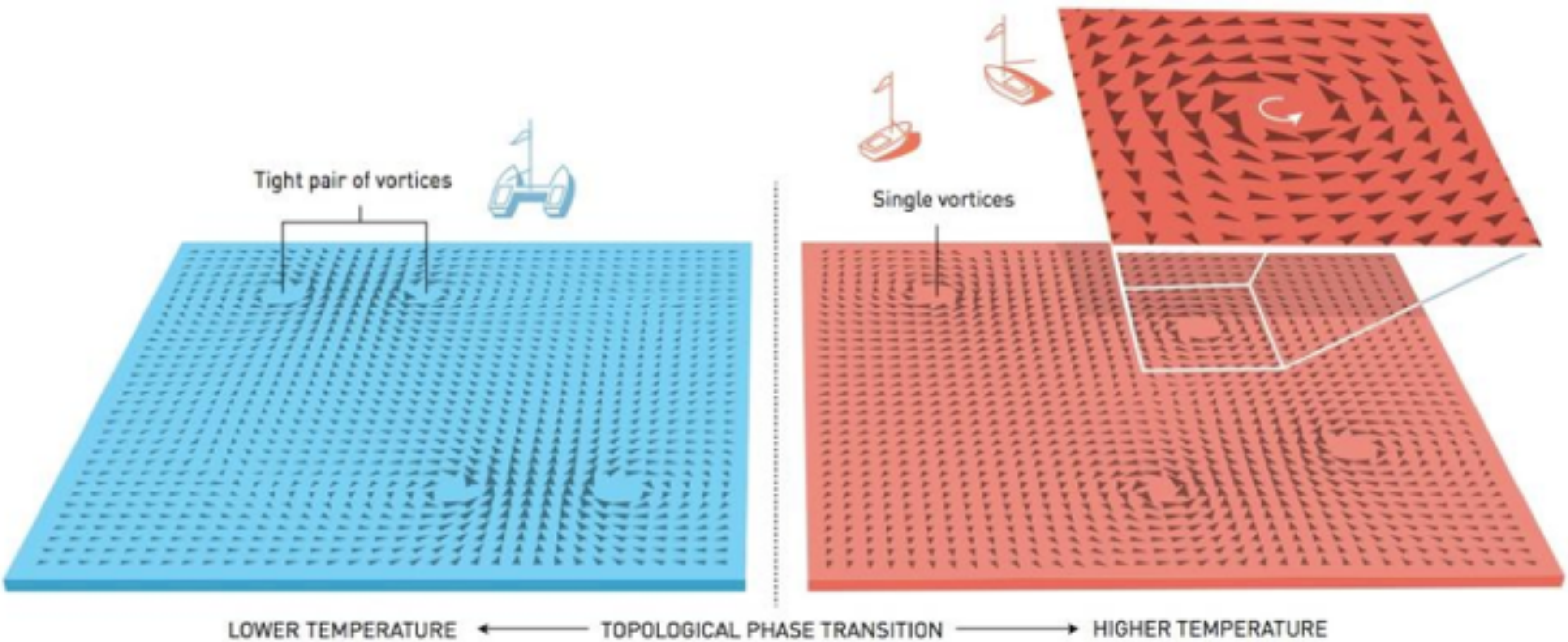
Critical temperature $\quad F = 0 \rightarrow T = \dfrac{\pi J}{2k_B}$

At $T < T_c$

Tightly bound pairs of vortex/antivortex



At $T = T_c$ the pairs break and the vortices and antivortices become free

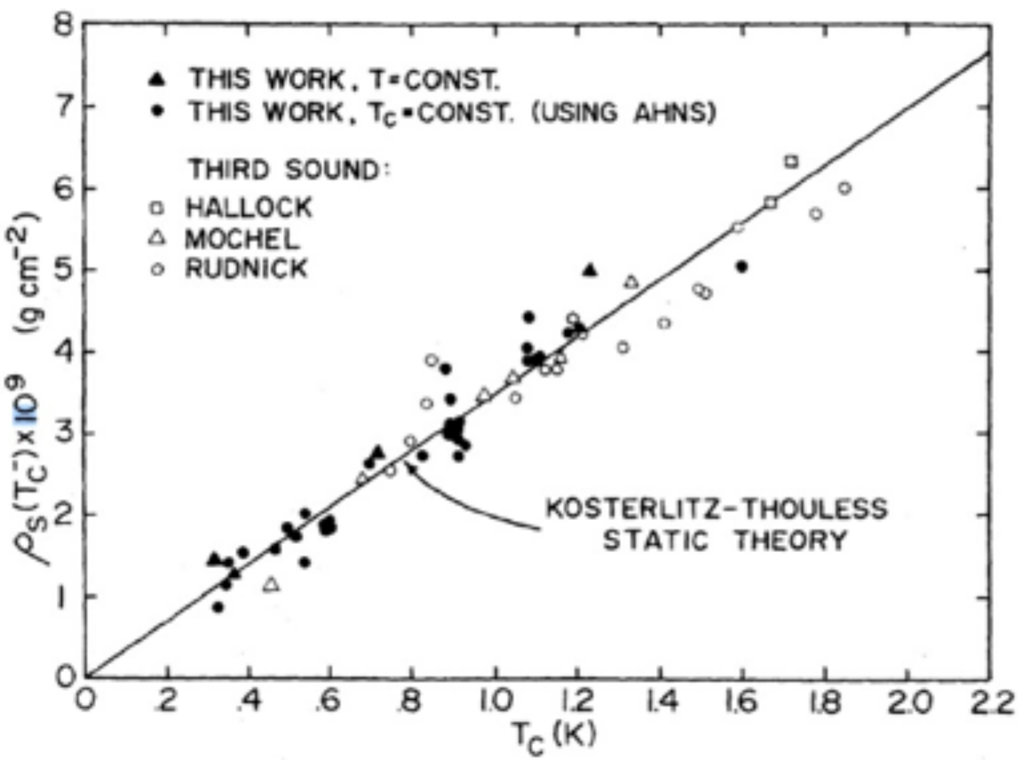


Experimental confirmations

Nelson and Kosterlitz (1977) : universal relation for films of superfluid ^{4}He

$$\lim_{T \to T_c^-} \frac{\rho_s(T)}{T} = \frac{2 m^2 k_B}{\pi \hbar^2}$$

Bishop and Reppy (1978) : experimental confirmation



Vadim L'vovich Berezinskii, 1935 (Kiev), 1980 (Moscow)

SOVIET PHYSICS JETP VOLUME 32, NUMBER 3 MARCH, 1971

DESTRUCTION OF LONG-RANGE ORDER IN ONE-DIMENSIONAL AND TWO-DIMENSIONAL SYSTEMS HAVING A CONTINUOUS SYMMETRY GROUP I. CLASSICAL SYSTEMS

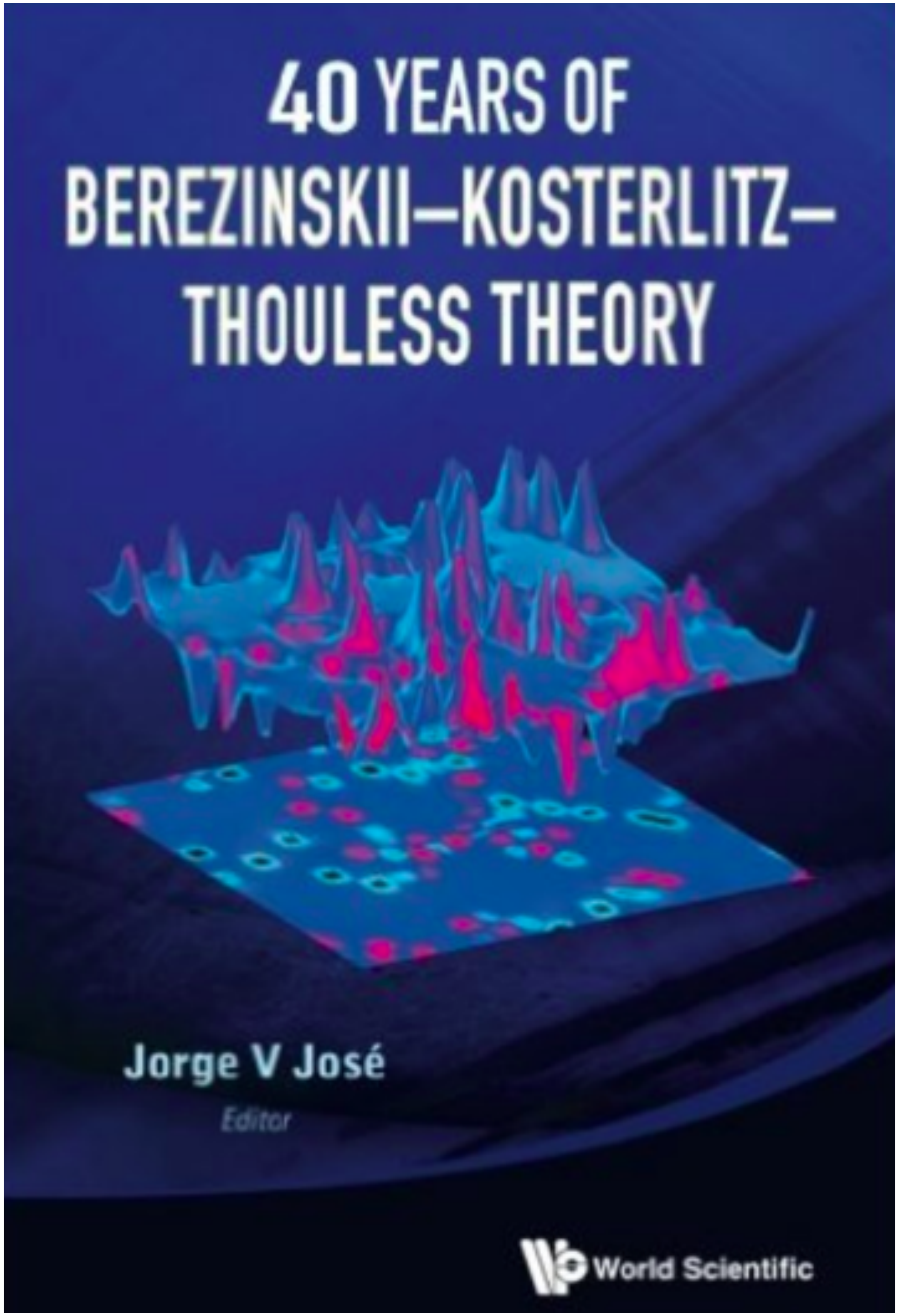
V. L. BEREZINSKIĬ

Submitted March 31, 1970

Zh. Eksp. Teor. Fiz. 59, 907–920 (September, 1970)

The low-temperature state of two-dimensional classical systems, which in the three-dimensional case have an ordered phase with a spontaneous violation of a continuous symmetry (magnetic substances, crystals), is considered. It is shown that for arbitrary dimension the long-range correlations are determined by an expression for the energy of the long wavelength fluctuations, which is quadratic with respect to the gradients. The distinctive feature of the one- and two-dimensional cases is that the fluctuation deflections grow with distance and at sufficiently large distances may reach a finite value, which leads to the necessity to take account of the effects associated with these. Thus, for a lattice of plane classical spins (Sec. 1) the contribution from configurations, where the spin vector on a path between sufficiently distant points is turned through an angle containing several complete revolutions, becomes essential.

KT transition -> BKT transition



Published in 2013

$\frac{1}{4}$ *Prize*

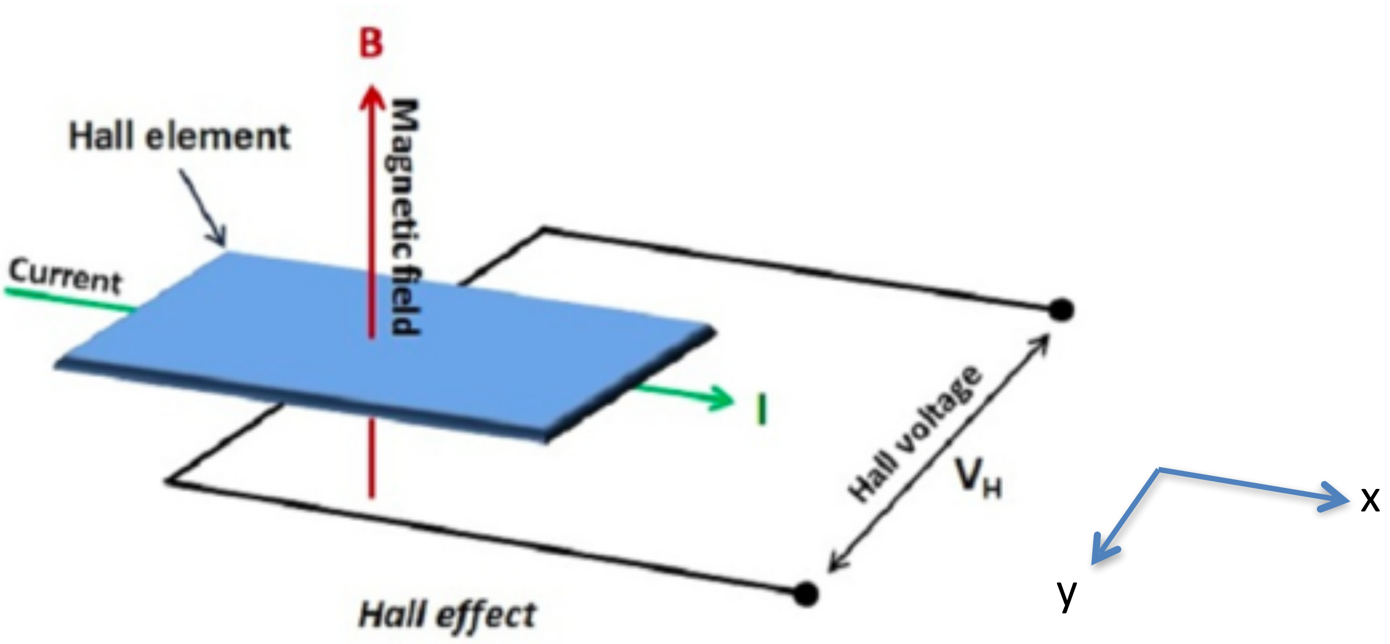
Hall effect

Edwin H. Hall
1855-1938

Electron gas in a plane subject to a perpendicular magnetic field

Appears of a voltage V_y perpendicular to the applied current I_x

Hall conductance $\sigma_{xy} = \frac{I_x}{V_y}$

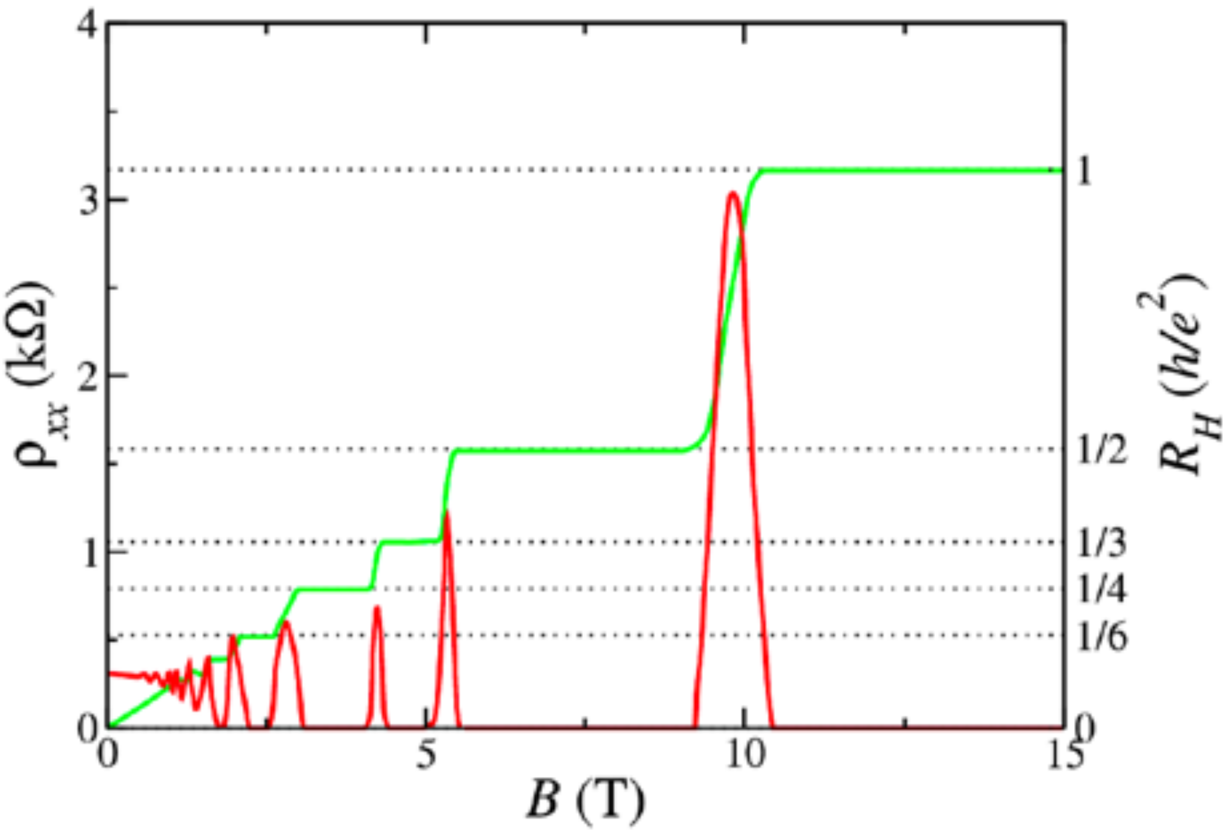


Integer quantum Hall effect

At low temperatures (< 2 K) and high magnetic fields (~ 10T)

$$\sigma_{xy} = n\frac{e^2}{h}, \quad n = 1,2,\ldots$$

Klaus von Klintzing
Nobel Physics 1985



Precision $\delta R_H / R_H = O(10^{-9})$

Independent of the sample, material, geometry, impurities

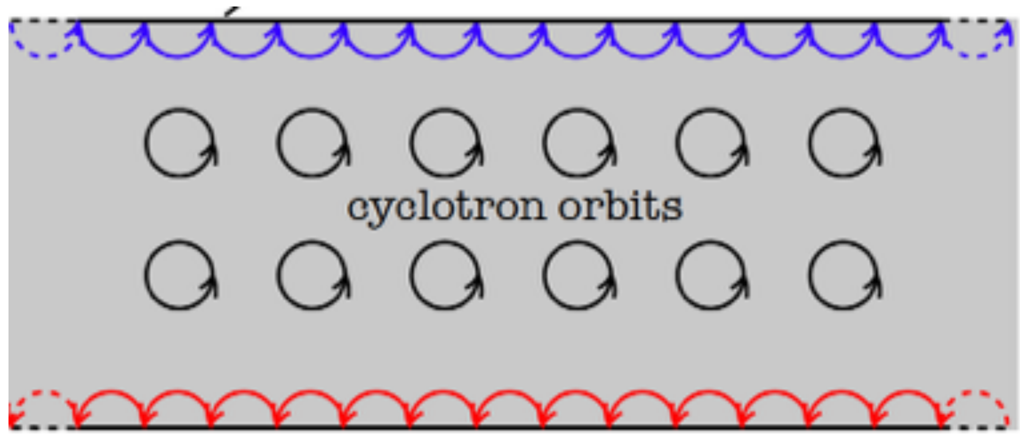
Universal property but also Topological ?

Von Klitzing constant $R_K = \dfrac{h}{e^2} = 25812.807557(18)\,\Omega$

Standard of resistance (SI)

Related to the fine structure constant $\alpha = \dfrac{e^2}{\hbar c} \approx 1/137$

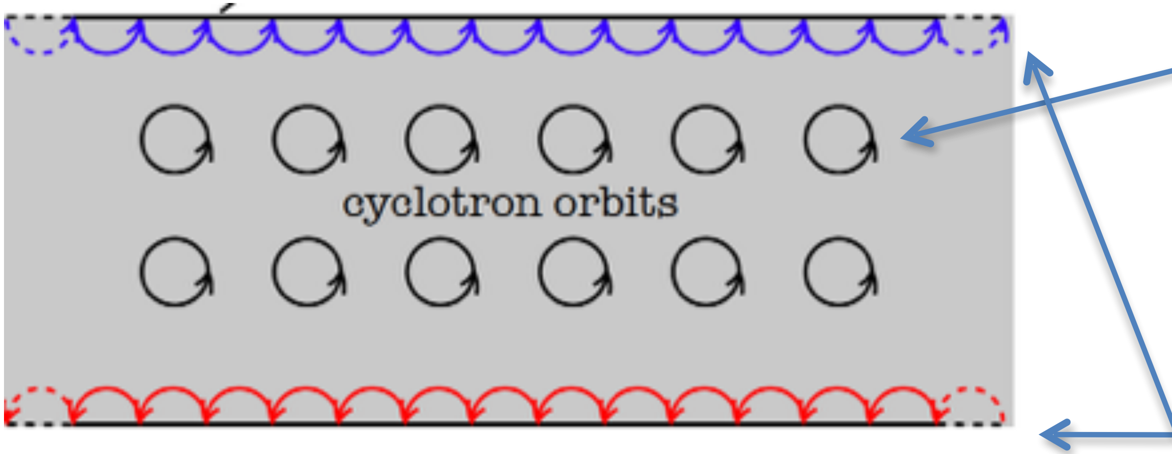
Origin of Integer Quantum Hall



Quantization of cyclotron orbits → Landau Levels

$$E_n = \hbar\omega_c (n + 1/2), \quad n = 0,1,\ldots \qquad \omega_c = \frac{eB}{mc}$$

Filling the n lowest Landau levels gives $\sigma_{xy} = n\frac{e^2}{h}, \quad n = 1,2,\ldots$



No transport of charge in the bulk:

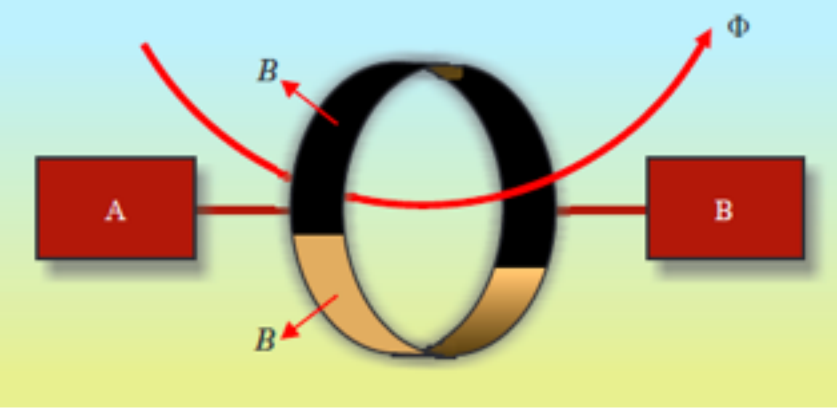
Bulk Insulator

Transport of charge at the edge:

Edge metal

R. Laughlin 1981

Recognize the existence of edge modes
Based on gauge invariance to explain the exactness of the quantized Hall conductance



Pumping electrons with flux

VOLUME 49, NUMBER 6 PHYSICAL REVIEW LETTERS 9 AUGUST 1982

Quantized Hall Conductance in a Two-Dimensional Periodic Potential

D. J. Thouless, M. Kohmoto,[a] M. P. Nightingale, and M. den Nijs
Department of Physics, University of Washington, Seattle, Washington 98195
(Received 30 April 1982)

The Hall conductance of a two-dimensional electron gas has been studied in a uniform magnetic field and a periodic substrate potential U. The Kubo formula is written in a form that makes apparent the quantization when the Fermi energy lies in a gap. Explicit expressions have been obtained for the Hall conductance for both large and small $U/\hbar\omega_c$.

$$\sigma_H = n\frac{e^2}{h}, \quad n = 1,2,\ldots$$ n is topological number

Topological meaning of Hall conductance

Wave functions of the Landau levels $u_{\vec{k},n}(\vec{r})$

$\vec{k}$: crystal momentum $\qquad$ n: Landau level

Kubo formula $$\sigma_H = \frac{e^2}{2\pi\hbar}\sum_n \int_{\vec{k}\in BZ} d^2k\, \mathrm{B}(\vec{k},n)$$

Berry potential $$\mathrm{A}_j(\vec{k},n) = \left\langle u_{\vec{k},n} \middle| \partial_{k_j} \middle| u_{\vec{k},n} \right\rangle$$

Berry curvature $$\mathrm{B}(\vec{k},n) = \partial_{k_x}\mathrm{A}_y(\vec{k},n) - \partial_{k_y}\mathrm{A}_x(\vec{k},n)$$

Chern number $$\frac{1}{2\pi}\int_{\vec{k}\in BZ} d^2k\, \mathrm{B}(\vec{k},n) = C_1(n) \in Z$$

Filled Landau levels $$C_1(n) = 1 \;\rightarrow\; \sigma_H = \frac{e^2}{h} n$$

Gauss- Bonet formula

$$\frac{1}{2\pi}\int_{\Sigma_g} d^2x\ R = \chi = 2(1-g)$$

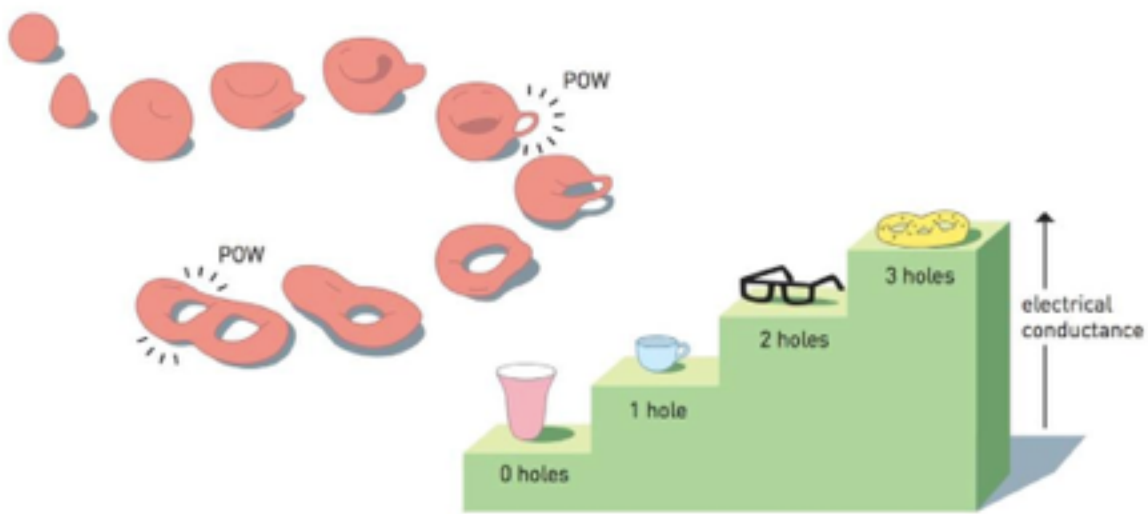
Curvature of surface

Number of holes

Chern formula

$$\frac{1}{2\pi}\int_{\vec{k}\in BZ} d^2k\ \mathrm{B}(\vec{k},n) = C_1(n)$$

Berry curvature

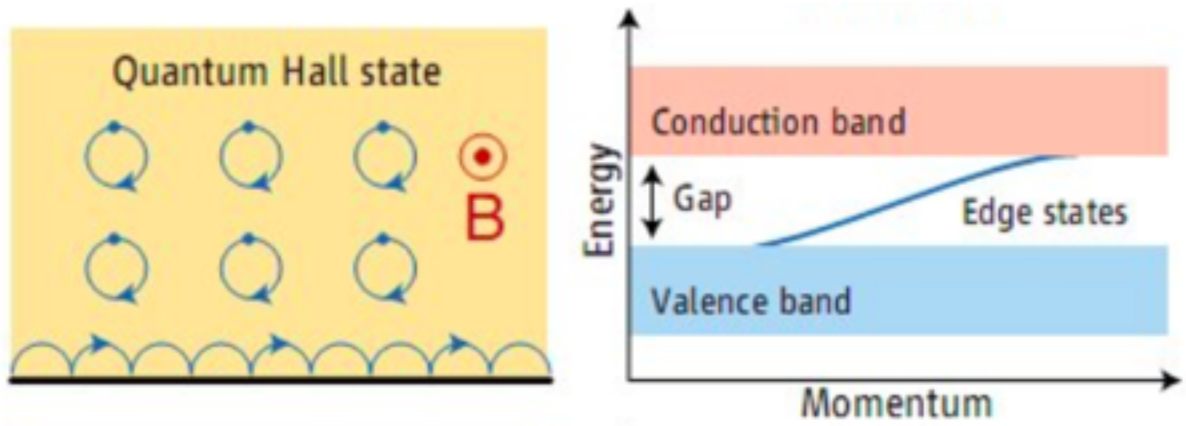


Meaning of Chern number

In the bulk The Block wave function in k-space has vortices

C_1 is the total vorticity

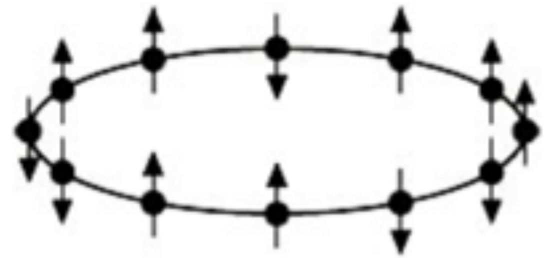
At the edge C_1 is the number of chiral edges modes (or antichiral)



1/4 *Prize*

Quantum Magnetism in lineland

$$\vec{S}_n = \frac{1}{2}\vec{\sigma}_n \rightarrow$$

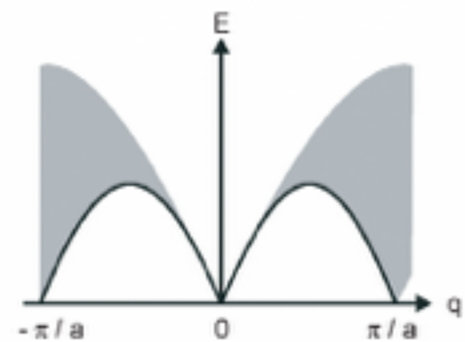


H. Bethe
1906 - 2005

$$H = \frac{J}{4}\sum_{n=1}^{N}\vec{\sigma}_n \cdot \vec{\sigma}_{n+1}$$

Bethe exact solution $$H|\psi\rangle = E|\psi\rangle$$

- Quasi long range $$\langle \vec{S}_0 \cdot \vec{S}_n \rangle \propto \frac{(-1)^n}{|n|}, \quad n >> 1$$
- No gap in the excitation spectrum



Nonlinear Field Theory of Large-Spin Heisenberg Antiferromagnets: Semiclassically Quantized Solitons of the One-Dimensional Easy-Axis Néel State

F. D. M. Haldane

Department of Physics, University of Southern California, Los Angeles, California 90089

(Received 31 January 1983)

The continuum field theory describing the low-energy dynamics of the large-spin one-dimensional Heisenberg antiferromagnet is found to be the O(3) nonlinear sigma model. When weak easy-axis anisotropy is present, soliton solutions of the equations of motion are obtained and semiclassically quantized. Integer and half-integer spin systems are distinguished.

Haldane Conjecture: there is a gap in the spectrum if the spin is integer

$\vec{S}_n$ spin S=1,2,...

$$H = J \sum_n \vec{S}_n \cdot \vec{S}_{n+1}, \quad J > 0$$

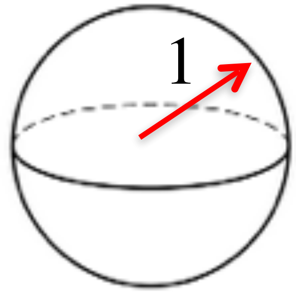
$$\Delta = \min(E_{exc}) - E_{GS} > 0$$

Finite correlation length ξ

$$\left\langle \vec{S}_0 \cdot \vec{S}_n \right\rangle \propto (-1)^n e^{-n/\xi}, \quad n >> 1$$

Map : Low energy of the spin chain to the Non Linear Sigma Model (NLSM)

$$\vec{\varphi}(x,t) \in$$



In the classical limit $S >> 1$ Haldane obtained the effective action

$$S = \frac{1}{g}\int dt\, dx \left[\frac{1}{v}\left(\partial_t \vec{\varphi}\right)^2 - v\left(\partial_x \vec{\varphi}\right)^2\right] \qquad g = \frac{2}{S} \to 0$$

NLSM : toy model of QCD : asymptotic freedom (Polyakov 1975)

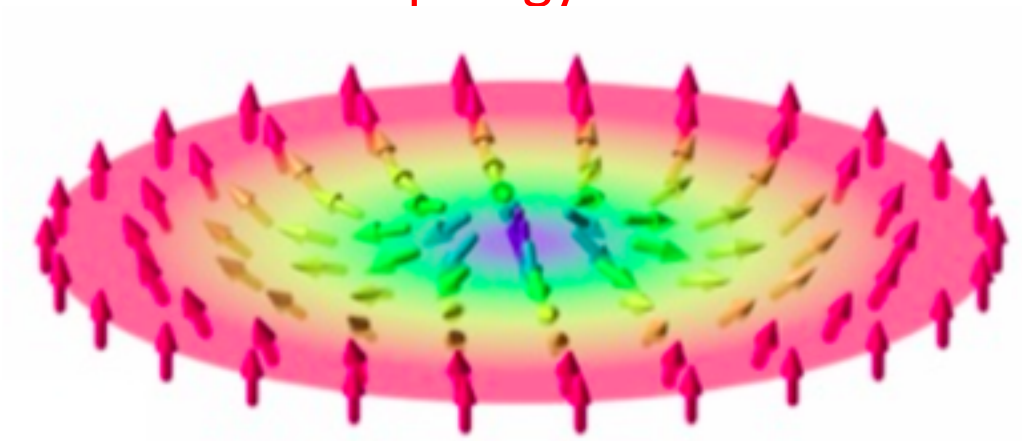
RG flow: $0 \xleftarrow{\text{UV}} g(\lambda) \xrightarrow{\text{IR}} \infty$

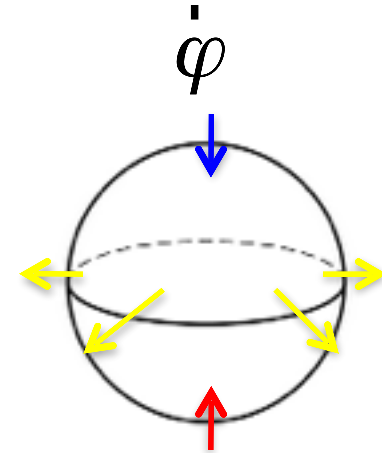
dynamical generation of mass

$$mass \propto \Lambda e^{-2\pi/g} \;\rightarrow\; gap \propto e^{-\pi S}$$

Polyakov

Topology in the NLSM





Contribute to the Euclidean action $$S_{top} = i\vartheta \int \frac{d^2x}{4\pi} \vec{\varphi} \left(\partial_1 \vec{\varphi} \times \partial_2 \vec{\varphi} \right) \qquad x^1 = it, x^2 = x$$

Winding number $$Q = \int \frac{d^2x}{4\pi} \vec{\varphi} \left(\partial_1 \vec{\varphi} \times \partial_2 \vec{\varphi} \right) \quad \text{: integer} \quad (Q = 1)$$

Haldane map $$\vartheta = 2\pi S$$

Partition function $$Z = \int D\vec{\varphi}\, e^{-S_{NSLM} - S_{top}} = \int D\vec{\varphi}\, e^{-S_{NSLM} - 2\pi S Q i}$$

$$e^{-2\pi S Q i} = 1 \;\; (S = 1, 2, \ldots), \quad e^{-2\pi S Q i} = (-1)^Q \;\; (S = 1/2, 3/2, \ldots)$$

PHYSICAL REVIEW B VOLUME 48, NUMBER 6 1 AUGUST 1993-II

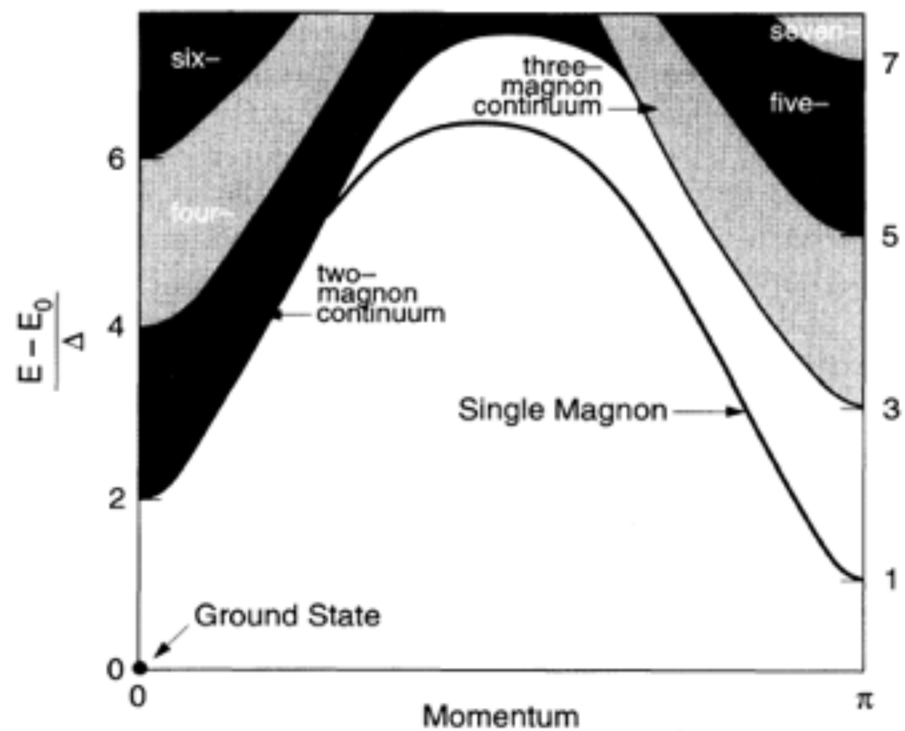
Numerical renormalization-group study of low-lying eigenstates of the antiferromagnetic $S = 1$ Heisenberg chain

Steven R. White
Department of Physics, University of California, Irvine, California 92717

David A. Huse
AT&T Bell Labs, Murray Hill, New Jersey 07974

Using the Density Matrix Renormalization Group (DMRG)

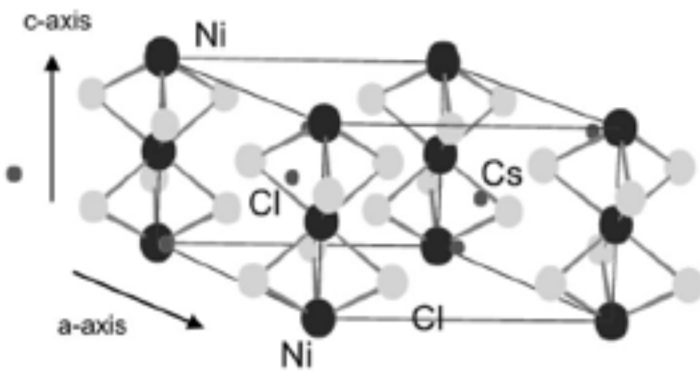
$$\Delta = 0.41050(2),$$
$$\xi = 6.03(1)$$

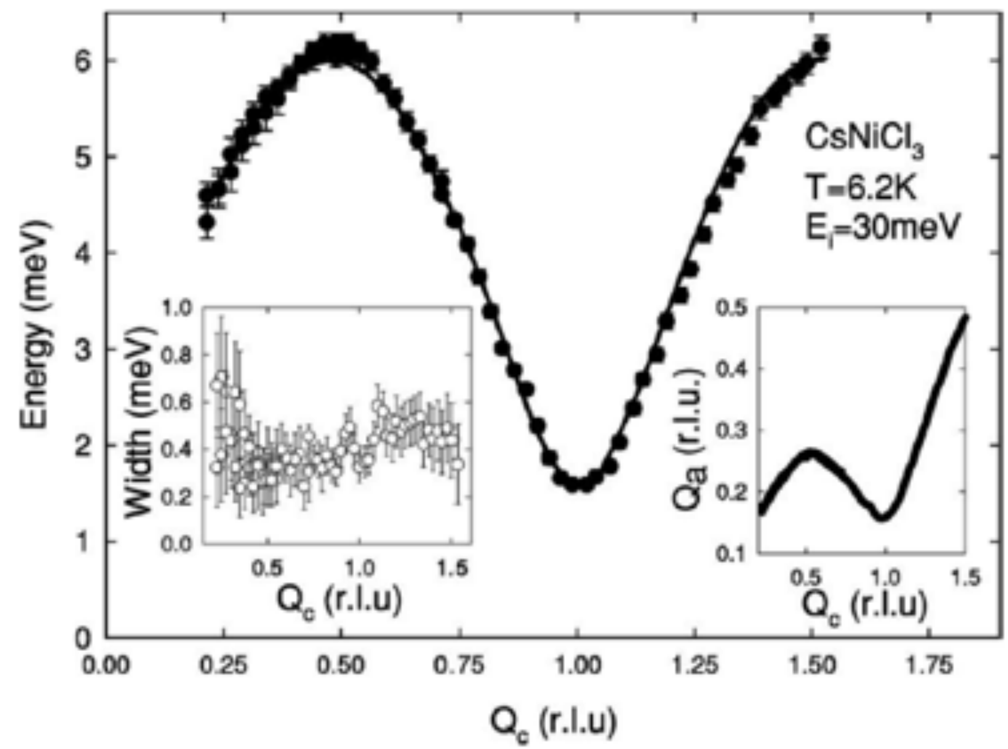


PHYSICAL REVIEW B 66, 024407 (2002)

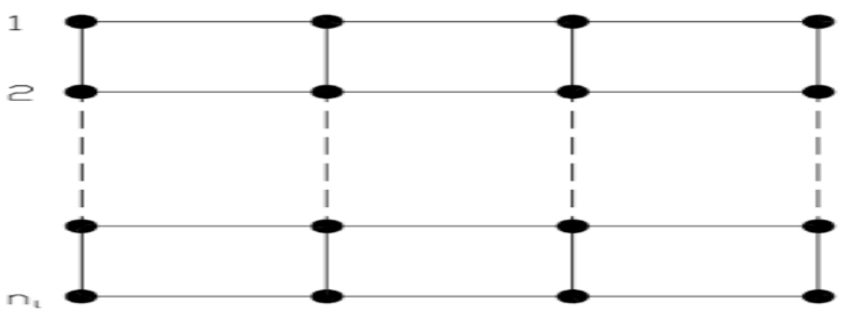
Properties of Haldane excitations and multiparticle states in the antiferromagnetic spin-1 chain compound $CsNiCl_3$

M. Kenzelmann,[1] R. A. Cowley,[1] W. J. L. Buyers,[2,3] Z. Tun,[2] R. Coldea,[4,5] and M. Enderle[6,7]





Spin ladders : n coupled spin ½ chains



$$\vartheta = \pi n$$

$$n : even \rightarrow gapped$$

$$n : odd \rightarrow gapless$$

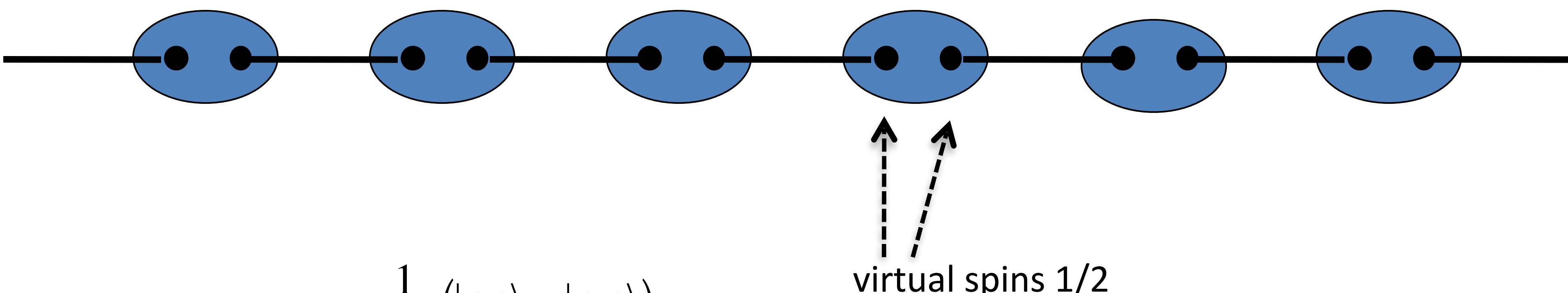
VOLUME 59, NUMBER 7 PHYSICAL REVIEW LETTERS 17 AUGUST 1987

Rigorous Results on Valence-Bond Ground States in Antiferromagnets

Ian Affleck,[a] Tom Kennedy, Elliott H. Lieb, and Hal Tasaki

$$H = \sum_i \left[\vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{3} (\vec{S}_i \cdot \vec{S}_{i+1})^2 \right]$$

Exact ground state

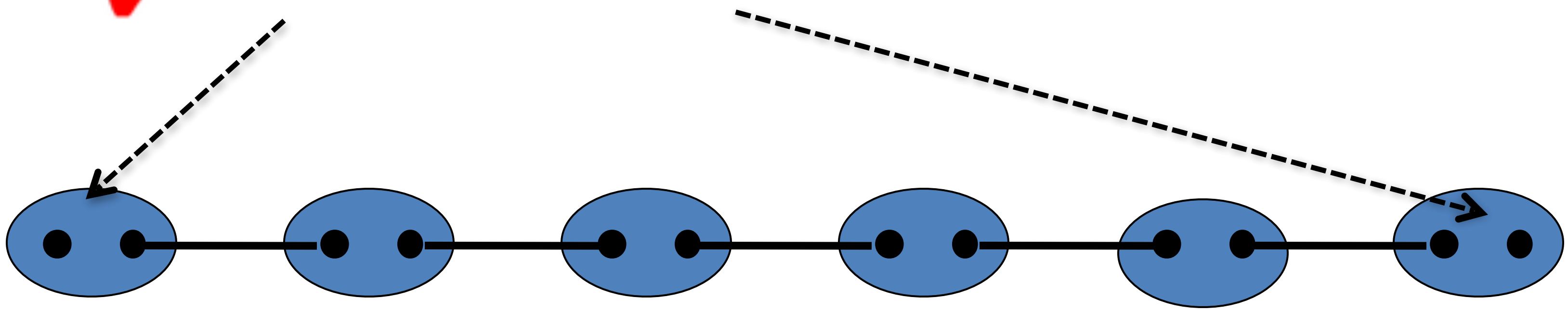


 $= \frac{1}{\sqrt{2}} \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right)$

✓ There is a Haldane gap

✓ $$\left\langle \vec{S}_0 \cdot \vec{S}_n \right\rangle = 4(-1)^n\, 3^{-n}, \quad \forall\, n$$

✓ unpaired edge spins ½



Ground state degeneracy = 4

Topological matter

Prototype in 2D : Quantum Hall and in 1D : Haldane chain

- ✓ Unbroken symmetries (failure of Landau paradigm)
- ✓ Gap of the bulk excitations
- ✓ Edge modes: gapless in 2D and localized in 1D
- ✓ Degenerate ground states that are indistinguishable locally

God created the bulk, surfaces where invented by the devil

W. Pauli

New Topological Matter

Majorana wires

(Kitaev 2000)

1D superconductor

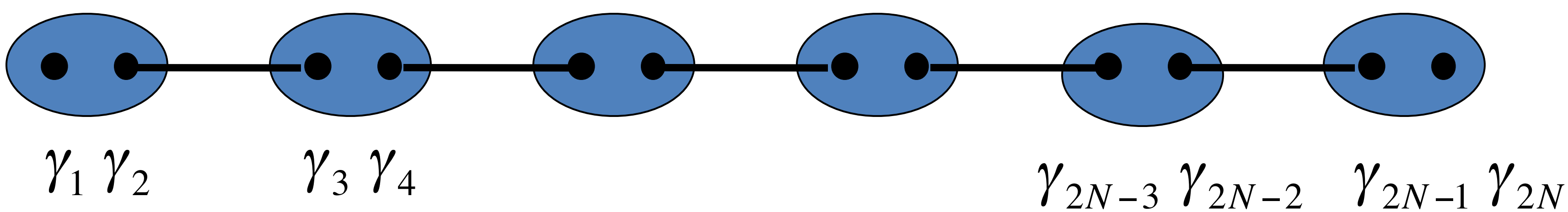
$$H = \sum_{j=1}^{N-1} t\, c^*_j c_{j+1} + \Delta\, c_j c_{j+1} + h.c.$$

Kinetic energy   Pairing interaction

Choosing

$$t = |\Delta| \quad , \quad c_j = \frac{1}{\sqrt{2}}\left(\gamma_{2j-1} + i\,\gamma_{2j}\right)$$

$$H = it\left(\gamma_2\gamma_3 + \gamma_4\gamma_5 + \ldots + \gamma_{2N-2}\gamma_{2N-1}\right)$$

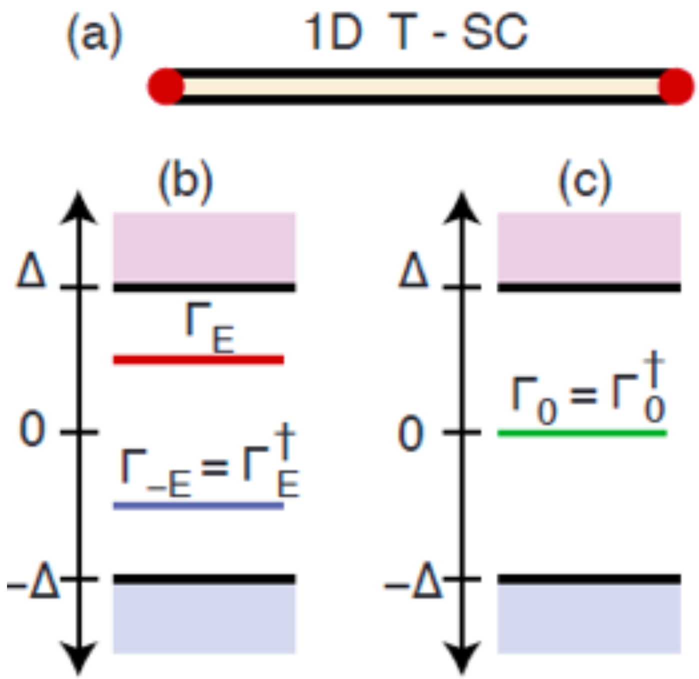


$\gamma_1,\ \gamma_{2N}$ Majorana edge modes

1D Topological superconductor

Topological Superconductors

Kitaev wire (2000)



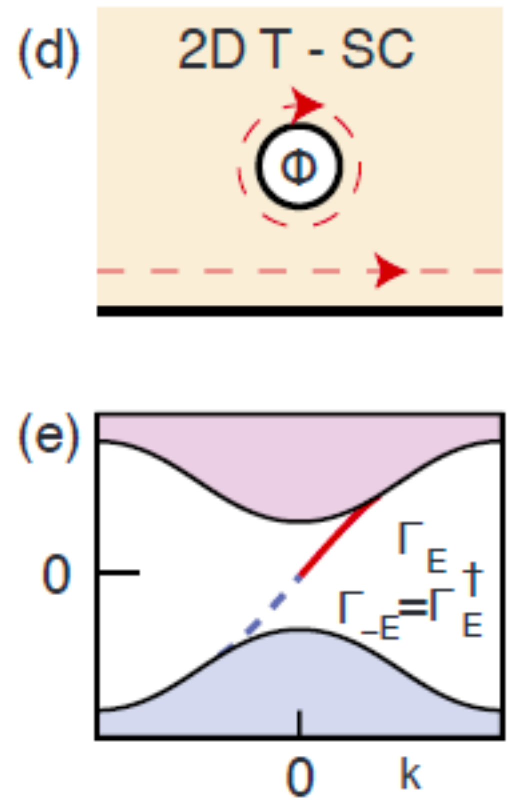
Trivial SC

Non trivial SC

Majorana modes
at the edges

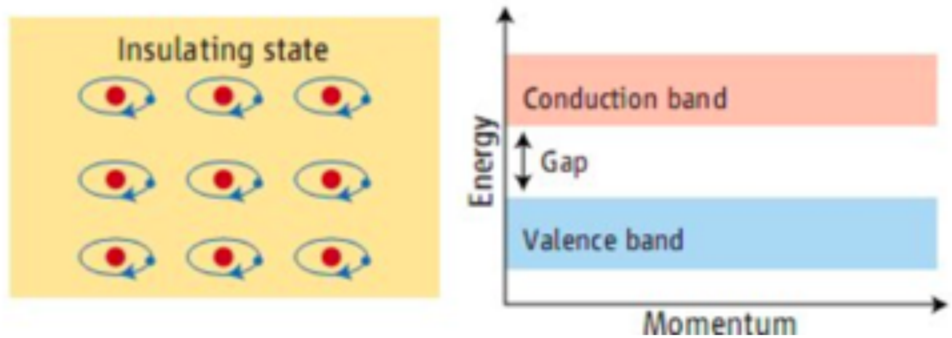
$p_x + i\,p_y$ symmetry

Read, Green
(2000)

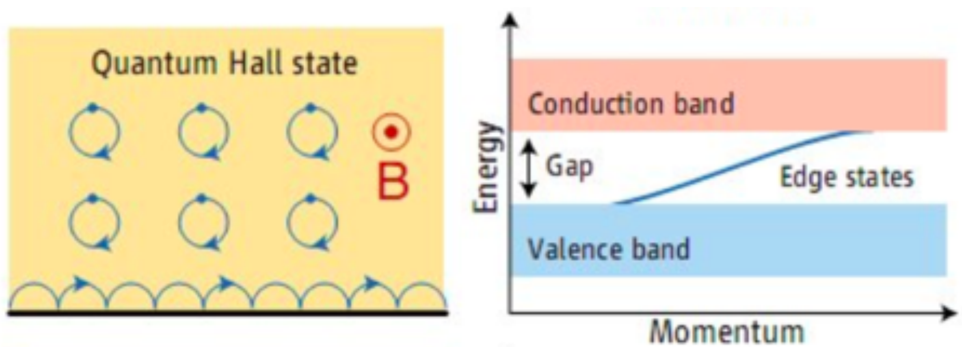


Majorana modes
around the vortices

Topological Insulators

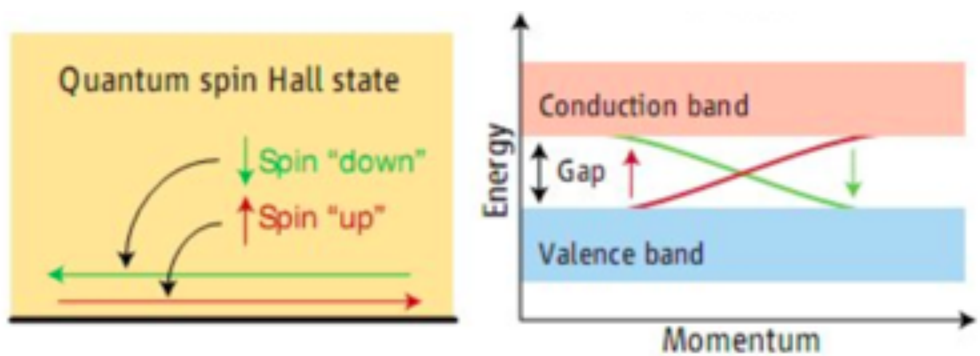


Trivial insulator



Haldane (1988):

B=0 but ***T*** broken



Kane and Mele (2005)
topological insulator
T is preserved

Periodic Table Topological Insulators and Superconductors

Θ = time reversal $\quad$ Ξ = particle-hole $\quad$ $\Pi = \Xi\Theta$ = chiral symmetry

d = dimension of space

AZ	Θ	Ξ	Π	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	−1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	−1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	−1	−1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	−1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	−1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

= Quantum Hall $\quad$ = Topological insulators $\quad$ = Topological superconductors

= Superfluid $^3He\,B$

Topological phases/ Topological matter

Thouless et al (1982), Wen (1995), Kitaev (2000),…

- No symmetry breaking (~~Landau paradigm~~)

- Stability under changes of materials, perturbations, etc

- Fractionalization of quantum numbers and/or exotic excitations

- Dependence on the topology in real space or momentum space

- Degeneracy of ground states that are locally indistinguishable

- Bulk/edge correspondence

- Symmetry protected phases

- Physical properties quantified by topological invariants in Maths

- Applications: Spintronics and Quantum Computation

References

- Scientific Background on the Nobel Prize in Physics 2016

- Colloquium : Topological insulators, Hasan and Kane (RMP, 2010)

Thanks / Gracias