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GRAVITATIONAL ANOMALIES

Luis ALVAREZ-GAUMÉ[1]
Lyman Laboratory of Physics, Harvard University, Cambridge, MA 02138, USA

Edward WITTEN[2]
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544, USA

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It is shown that in certain parity-violating theories in $4k+2$ dimensions, general covariance is spoiled by anomalies at the one-loop level. This occurs when Weyl fermions of spin-$\frac{1}{2}$ or -$\frac{3}{2}$ or self-dual antisymmetric tensor fields are coupled to gravity. (For Dirac fermions there is no trouble.) The conditions for anomaly cancellation between fields of different spin is investigated. In six dimensions this occurs in certain theories with a fairly elaborate field content. In ten dimensions there is a unique theory with anomaly cancellation between fields of different spin. It is the chiral $n=2$ supergravity theory, which is the low-energy limit of one of the superstring theories. Beyond ten dimensions there is no way to cancel anomalies between fields of different spin.

1. Introduction

The fermion anomaly in $(3+1)$-dimensional quantum field theory has a remarkable number of important applications. In the original version [1], one considers a massless fermion triangle diagram with one axial current and two vector currents. Requiring conservation of the vector currents, one finds, even for massless fermions, that the axial current is not conserved (fig. 1). This results in a breakdown of chiral symmetry in the presence of gauge fields that are coupled to the conserved vector currents. This breakdown is known to lead to an understanding of π^0 decay and to a resolution of the U(1) problem in QCD [2].

Another, equally significant facet of the anomaly arises if gauge fields are coupled not to vector currents but to linear combinations of vector and axial vector currents, as in the standard $\mathrm{SU}(2)\times\mathrm{U}(1)$ model of weak interactions. For instance, in a gauge theory with $\mathrm{V}-\mathrm{A}$ gauge couplings, one must consider (fig. 2) a fermion triangle with a $\mathrm{V}-\mathrm{A}$ current at each vertex. This diagram is again anomalous. Unless it cancels when summing over the fermion species running around the loop, the anomaly spoils conservation of the $\mathrm{V}-\mathrm{A}$ currents. But gauge theories with gauge fields coupled to non-conserved currents are inconsistent. So the $\mathrm{SU}(2)\times\mathrm{U}(1)$ model (or any gauge theory with non-vectorlike gauge couplings) is inconsistent unless the

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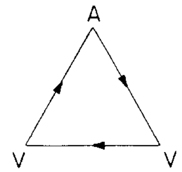

Fig. 1. The fermion triangle with one axial current and two vector currents.

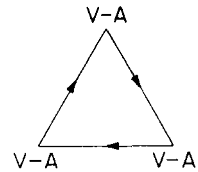

Fig. 2. The triangle diagram with a V − A insertion at each vertex.

anomalies cancel [3]. The classic triangle anomaly has other important applications as well [4].

These remarks can be generalized in various directions. First of all, in addition to the fermion triangle diagram with three currents, the triangle with one current and two energy-momentum tensors is anomalous (fig. 3) [5]. This has been widely discussed in connection with the breakdown of chirality conservation in a gravitational field [6].

Also, several authors have recently discussed the question of anomalies in space–times of higher dimension. In any even number of dimensions, there is an anomaly analogous to the triangle anomaly. In N dimensions, a fermion one-loop diagram with $\frac{1}{2}N+1$ external gluons is potentially anomalous. The anomaly can be evaluated [7] by analogy with the usual evaluation of the triangle in four dimensions. Cancellation of anomalies in higher dimensions is potentially of interest as a restriction on Kaluza–Klein theories. For instance, as noted by some of the authors of ref. [7], the supersymmetric Yang–Mills theory in ten dimensions is inconsistent because of hexagon anomalies. This problem is even more serious than the lack of renormalizability (which has been explicitly demonstrated [8]) because it represents a breakdown of gauge invariance which almost certainly cannot be cured by a short distance cut-off. This point will be developed in sect. 3; it is related to 't Hooft's observation [4] that anomalies can be understood in terms of low-energy physics.

The thrust of the present paper is the following. The need to cancel anomalies places restrictions on which theories can be coupled to gravity. Such a restriction arises in four dimensions as a straightforward consequence of the triangle anomaly of fig. 3. Specifically, the SU(2) × U(1) theory cannot be coupled to gravity unless the sum of the hypercharges of the left-handed fermions vanishes. This appears to

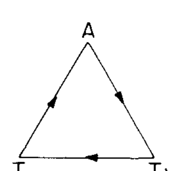

Fig. 3. The anomalous triangle with one axial current A and two insertions of the energy-momentum tensor.

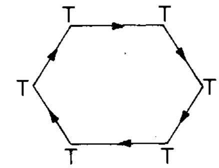

Fig. 4. The hexagon anomaly in ten dimensions; a diagram with six insertions of the energy-momentum tensor T.

be the case for known fermions (it is true for each generation of quarks and leptons). This will be further discussed in sect. 3.

Our main effort, however, will be devoted to unraveling the structure of anomalies in more than four dimensions. As we will see, in certain theories there are one-loop anomalies in the purely gravitational interactions. This sort of anomaly does not occur in four dimensions, and this is probably why it has not been previously noted*. Purely gravitational anomalies occur for Weyl fermions (of spin $\frac{1}{2}$ or spin $\frac{3}{2}$) in $N=4k+2$ dimensions, $k=0,1,2,\ldots$. For instance, in ten dimensions (fig. 4) the hexagon diagram with six external gravitons coupled to a Weyl fermion of definite chirality is anomalous. It is impossible to maintain Bose symmetry and gauge invariance for the external graviton lines. This anomaly arises for Weyl fermions of spin $\frac{1}{2}$ or spin $\frac{3}{2}$. We will find a similar anomaly in the gravitational coupling of self-dual antisymmetric tensor Bose fields. Because of these anomalies, Weyl fermions or self-dual tensors cannot be coupled to gravity in $4k+2$ dimensions, unless one arranges for anomaly cancellation between the different spins.

A number of interesting theories are affected by this discussion. For instance, in ten dimensions, there is an $n=1$ supergravity theory with one real Weyl gravitino (and one real spin-$\frac{1}{2}$ field of opposite chirality). This theory is anomalous.

In ten dimensions there are two theories with $n=2$ supergravity. One can be obtained by dimensional reduction from eleven-dimensional supergravity [10]. It is parity conserving, and like all parity conserving theories it is anomaly free. The other theory [11] has two gravitinos of the same chirality. It has one-loop anomalies in the gravitino, spin-$\frac{1}{2}$, and antisymmetric tensor couplings, but as we will see the anomalies of the fields of different spin cancel. This is, in fact, apparently the only ten-dimensional theory with anomaly cancellation between fields of different spin.

Since anomalies can be understood in terms of the *low*-energy behavior of a theory [4], the same considerations apparently apply to the supersymmetric string theories [12] which are of much interest as an approach to quantum gravity. The "type (I)" superstring theories reduce at low energy to $n=1$ supergravity in ten

* There is a well-known trace anomaly in four dimensions [9]. This concerns the anomalous trace of the energy-momentum tensor and is related to the Callan–Symanzik β function. It does not spoil the *conservation* of the energy-momentum tensor and does not ruin the mathematical consistency of coupling to gravity.

dimensions. They are presumably anomalous. (Introduction of a gauge group cannot remove the gravitational anomaly, as one may see from formulae of sect. 12. On the contrary [7] it introduces new anomalies.) The "type (II)" theories of closed superstrings only reduce at low energies to $n=2$ supergravity in ten dimensions. They appear to be free of hexagon anomalies. (Unfortunately, it is difficult to check these statements directly in the string context, because the simple light-cone formalism [12] is restricted to external graviton lines of $p^+=\varepsilon^{+\mu}=0$, for which the kinematical factor R defined in sect. 6 vanishes. Of course, a string theory might have anomalies that disappear in the field theoretic limit and survive when $p^+=\varepsilon^{+\mu}=0$, but this is outside the scope of our investigation.)

In addition to the purely gravitational anomalies, there are (as in four dimensions) anomalies in one-loop diagrams with both external gluons and external gravitons. In general, in n dimensions, the one-loop diagrams with $2r$ external gluons and $\frac{1}{2}n+1-2r$ external gravitons is anomalous for $0 \leqslant r \leqslant \frac{1}{4}(n+2)$. For $r=\frac{1}{4}(n+2)$ this has been discussed previously [7]. The cancellation of all these anomalies is a very severe restriction on the allowed fermion quantum numbers in Kaluza–Klein theory. The phenomenology of some of the anomaly-free theories will be discussed elsewhere [13].

The construction of effective actions in curved space–time has been reviewed in ref. [14]. Our results answer in the negative the question of whether the induced energy-momentum tensor for matter fields in a background geometry is always conserved.

Our calculations will reveal a connection between the gravitational anomalies in $4k+2$ dimensions and index theorems on curved manifolds in $4k+4$ dimensions. This connection was suggested by M.F. Atiyah (private communication) on the basis of properties of diffeomorphism groups. Anticipating this connection was of considerable help in finding ways to calculate gravitational anomalies.

2. Some generalities about anomalies

Before considering particular theories in detail, let us begin with some generalities about anomalies. Most of the remarks in this section are well known.

First of all, as noted in the original literature, the anomaly constitutes a breakdown of gauge invariance. Theories with anomalies are theories in which the effective action is not gauge invariant at the one-loop level. To be specific, consider in four-dimensional euclidean space a theory with gauge fields coupled to fermion fields in a complex representation of the gauge group. Let us denote the left- and right-handed fermions as ψ and $\bar{\chi}$ respectively (in euclidean space they are independent variables, not complex conjugates of each other; and we use different names to emphasize the fact they they are independent). The lagrangian for ψ and $\bar{\chi}$

interacting with gauge fields is

$$\mathscr{L}=\int \mathrm{d}^4x\,\bar{\chi}(i\partial\!\!\!/-\gamma^\mu\Sigma A_\mu^a\lambda^a)\left(\frac{1-\gamma_5}{2}\right)\psi\,, \tag{1}$$

where the λ^a, normalized so that $\mathrm{Tr}\,\lambda^a\lambda^b=2\delta^{ab}$, are the gauge group generators acting on the left-handed fermions. If we define $A_\mu=\frac{1}{2}\Sigma\lambda^a A_\mu^a$, then the variation of A_μ under an infinitesimal gauge transformation is $A_\mu\to A_\mu-D_\mu\varepsilon$. Consequently, any functional $\Gamma(A)$ changes under an infinitesimal gauge transformation as

$$\begin{aligned}\Gamma(A_\mu)&\to\Gamma(A_\mu-D_\mu\varepsilon)\\&=\Gamma(A_\mu)-\int\mathrm{d}^4x\,\mathrm{Tr}\,D_\mu\varepsilon(x)\frac{\delta\Gamma}{\delta A_\mu(x)}\\&=\Gamma(A_\mu)+\int\mathrm{d}^4x\,\mathrm{Tr}\,\varepsilon(x)D_\mu\frac{\delta\Gamma}{\delta A_\mu(x)}\,,\end{aligned}$$

so the generator of gauge transformations is $D_\mu\,\delta/\delta A_\mu^a$. Now, let Γ be the one-loop fermion effective action; thus, formally,

$$\exp[-\Gamma(A_\mu)]=\int\mathrm{d}\bar{\chi}\,\mathrm{d}\psi\,\exp\left\{-\int\mathrm{d}^4x\left[\bar{\chi}iD\!\!\!\!/\left(\frac{1-\gamma_5}{2}\right)\psi\right]\right\}. \tag{2}$$

One often writes Γ as the logarithm of the determinant of the Dirac operator [15]; we will discuss shortly the limits of this formulation.

Although Γ is naively gauge invariant, the statement of the anomaly is that the variation of Γ under a gauge transformation does not vanish; rather

$$D_\mu\frac{\delta\Gamma}{\delta A_\mu^a}=-\frac{i}{48\pi^2}\varepsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}\,\lambda^a[2\partial_\mu A_\nu\partial_\alpha A_\beta-i\partial_\mu(A_\nu A_\alpha A_\beta)]\,. \tag{3}$$

Thus, the anomaly represents a failure of gauge invariance.

The connection of this loss of gauge invariance with the failure of current conservation due to the anomaly is as follows. The fermion current induced by an applied gauge field is

$$J_\sigma^a=\left\langle\bar{\chi}\gamma_\sigma\lambda^a\left(\frac{1-\gamma_5}{2}\right)\psi\right\rangle^{A_\mu},$$

where the expectation value is to be taken in the background field A_μ. However, from eq. (2) it follows that $J_\mu^a=\delta\Gamma/\delta A_\mu^a$ so the (covariant) divergence of the current is

$$D_\mu J_\mu^a=D_\mu\frac{\delta}{\delta A_\mu^a}\Gamma\,, \tag{4}$$

and the loss of gauge invariance is equivalent to a failure of current conservation.

In gauge theories, unitarity and (in the case of renormalizable theories) renormalizability depend upon gauge invariance. Once gauge invariance is lost, these properties are also lost.

Now let us return to eq. (3). The factor of i on the right-hand side of eq. (3) is of fundamental importance. (This factor has nothing to do with thresholds or unitarity; it is required by CPT in euclidean space for a parity violating amplitude proportional to $\varepsilon_{\mu\nu\alpha\beta}$.) Because of this factor of i, only the imaginary part of Γ is not gauge invariant. The real part of Γ is perfectly gauge invariant. In terms of the fermion integral, which equals $e^{-\Gamma}$, this means that the *modulus* of the fermion integral is gauge invariant; it is the *phase* of the fermion integral that is not gauge invariant.

There is a simple reason for this. First of all, consider the case in which the fermions are in a real representation of the gauge group. Such fermions are real (anticommuting) variables. For such real variables, the fermion integral $e^{-\Gamma}$ is real. Moreover, theories in which the fermions are in a real representation of the gauge group can be regularized by the Pauli–Villars method. The existence of this gauge invariant regularization ensures that the fermion effective action is gauge invariant. For fermions in a real representation of the gauge group, the effective action is real and gauge invariant.

Now consider fermions in a complex representation Q. Such fermions are complex objects and the fermion integral is not necessarily real. On the contrary, as we have noted, the parity-violating amplitudes are imaginary. Gauge invariance forbids bare masses for fermions in a complex representation Q, and for this reason there is no evident way to regularize such theories. So the effective action $\Gamma(\mathrm{Q})$ for fermions in the representation Q is complex and not necessarily gauge invariant.

Now consider fermions in the complex conjugate representation $\bar{\mathrm{Q}}$. Such variables are complex conjugates of the ones we have just considered, and the Dirac action for fermions in the $\bar{\mathrm{Q}}$ representation is the complex conjugate of the action for fermions in the Q representation. So the effective action $\Gamma(\bar{\mathrm{Q}})$ is the complex conjugate of $\Gamma(\mathrm{Q})$. (Differently put, in passing from Q to $\bar{\mathrm{Q}}$, the parity-violating amplitudes, which are imaginary, change sign, so the action is complex conjugated.) Hence $\Gamma(\mathrm{Q})+\Gamma(\bar{\mathrm{Q}}) = 2\,\mathrm{Re}\,\Gamma(\mathrm{Q})$.

But $\Gamma(\mathrm{Q})+\Gamma(\bar{\mathrm{Q}})$ is the effective action for fermions in the real representation $\mathrm{Q}+\bar{\mathrm{Q}}$. Because of our previous remarks, this effective action is gauge invariant. So $\mathrm{Re}\,\Gamma(\mathrm{Q})$ is gauge invariant, and only $\mathrm{Im}\,\Gamma(\mathrm{Q})$ may suffer from anomalies. Anomalies in $\mathrm{Im}\,\Gamma(\mathrm{Q})$ do in fact show up in triangle diagrams.

A variant of this can occur if Q is a pseudoreal but not real representation of the gauge group. In this case, bare masses and Pauli–Villars regularization are again not possible so there is no guarantee that $e^{-\Gamma}$ is gauge invariant. However, if Q is pseudoreal, then $\mathrm{Q}\oplus\mathrm{Q}$ is a real representation, so the corresponding fermion integral, which is $e^{-2\Gamma}$, can be regularized and is gauge invariant. The fact that $e^{-2\Gamma}$ is gauge invariant means that $e^{-\Gamma}$ is gauge invariant in absolute value but perhaps not in sign. In certain cases [16], it is indeed impossible to define the sign in a gauge invariant way.

There is another point of view about this which may seem cumbersome but does help clarify why the phase of the fermion integral is potentially anomalous.

We are accustomed to thinking of the fermion integral as the determinant of the Dirac operator. For fermions in a real representation, we consider the hermitian eigenvalue problem $i\not{D}\psi_i = \lambda_i\psi_i$ and define $\det i\not{D}$ as the product of the λ_i. Since the λ_i are real, their product is real; this shows again that the fermion integral is real for fermions in a real representation. Of course, the product of the λ_i needs to be regularized. But this is easily accomplished. One may define $\det i\not{D} = \prod F(\lambda_i)$ where $F(\lambda_i)$ is a suitably chosen function such that $F(\lambda) = \lambda$ for small λ and $F(\lambda) \to 1$ as $\lambda \to \infty$.

For fermions in a complex representation, life is more subtle. Our action is of the general form

$$\mathcal{L} = \bar{\chi} i\not{D}\left(\frac{1-\gamma_5}{2}\right)\psi, \tag{5}$$

where the ψ are left-handed fermions in a complex representation Q, and the $\bar{\chi}$ are right-handed fermions in the complex conjugate representation $\bar{\mathrm{Q}}$.

Let V be the vector space of left-handed fermion fields in the Q representation. Given a vector space V and an operator M (not necessarily hermitian) mapping V into V, it is possible to define the determinant of M. In the usual way, one solves the eigenvalue problem $M\psi_i = \lambda_i\psi_i$ (or one defines the λ_i to be the diagonal elements when M is put in Jordan canonical form) and one defines $\det M = \prod \lambda_i$. In general, this determinant is a complex number.

We would like to define a determinant of the operator $D = \frac{1}{2} i\not{D}(1-\gamma_5)$. The problem is that D does not map V into itself; it maps V into $\tilde{\mathrm{V}}$, the vector space of *right-handed* fermions in the Q representation. Without further information, there is no way to define a determinant of an operator M that maps one vector space V into another space $\tilde{\mathrm{V}}$.

In the case at hand, V and $\tilde{\mathrm{V}}$ are Hilbert spaces; for ψ in V or $\tilde{\mathrm{V}}$, we define the norm of ψ as $\langle\psi|\psi\rangle = \int \mathrm{d}^4x \sum_{\alpha i} |\psi_{\alpha i}(x)|^2$. Given an operator M from one Hilbert space V to another Hilbert space $\tilde{\mathrm{V}}$, a determinant can be defined *up to phase* as follows. Let $|\psi_i\rangle$ be an orthonormal basis for V. Let $|x_i\rangle = M|\psi_i\rangle$. Choose the $|\psi_i\rangle$ so $\langle x_i|x_j\rangle = 0$ for $i \neq j$, and define $\lambda_i = \sqrt{\langle x_i|x_i\rangle}$. The product of the λ_i is automatically real and is not a sensible definition of $\det M$, since it does not reduce to the standard definition (which can be complex) when $\mathrm{V} = \tilde{\mathrm{V}}$. But it makes sense to adopt this definition of the *modulus* of the determinant: $|\det M| = \prod \lambda_i$.

What about the *phase* of $\det M$? Given a single operator M from V to $\tilde{\mathrm{V}}$, there is no way to define this phase. But suppose we have two operators M and N from V to $\tilde{\mathrm{V}}$. Then $A = N^{-1}M$ maps V into V, so the determinant of A is well defined as a complex number. The definition ln det M − ln det N = ln det A defines the phase *difference* between det M and det N.

In our physical problem, we do not care about the overall phase of the determinant of $D=\frac{1}{2}i\not{D}(1-\gamma_5)$, since a constant can be absorbed in normalizing the partition function. We do care about *relative* phases. We arbitrarily pick a convenient gauge field, say $A^a_\mu=0$, and define det D to be, say, positive for this gauge field. For any other gauge field A^a_μ one attempts to define the phase of the determinant by saying $\ln\det D(A^a_\mu)-\ln\det D(A^a_\mu=0)=\ln\det(D^{-1}(A^a_\mu=0)D(A^a_\mu))$. When dealing with differential operators in infinite-dimensional function spaces, it is difficult to define the determinant of an operator such as $D^{-1}(A^a_\mu=0)D(A^a_\mu)$. This difficulty is one way of understanding the origin of anomalies.

Let us now return to the Feynman diagrams in which anomalies appear. In four dimensions, the simplest fermion diagram which (kinematically) can violate parity is the diagram with three external gluons. The diagram is indeed anomalous, as we have already discussed. A few general remarks about diagrammatic evaluation of anomalies will be useful later.

There is no good way to regularize the anomalous diagrams; if there were, there would be no anomaly. Because there is no good way to regularize these diagrams, they are potentially ambiguous. The potential ambiguity consists of the ability to add a polynomial in the external momenta. The reason for this is as follows. Any acceptable definition of the triangle must obey unitarity in each channel. Using unitarity, the triangle amplitude (or any one-loop amplitude) can be uniquely reconstructed from tree diagrams up to a polynomial in the external momenta. (See recent discussions by Coleman and Grossman and by Frishman et al. [4].) Therefore, the triangle amplitude is well defined modulo the ability to add such a polynomial. When one claims that a diagram is anomalous, one means that it is impossible to add a polynomial in the momenta so as to eliminate the anomaly and obtain an amplitude that obeys all physical principles.

It automatically follows from this that, regardless of the superficial degree of divergence of a diagram, anomalies are always finite. After all, the infinite part of a diagram is always a polynomial in the external momenta. Our freedom to redefine an amplitude by adding a polynomial includes the freedom to throw away all infinite pieces. Hence, relevant anomalies, if any, are always finite.

Even in unrenormalizable theories, anomalies that ruin gauge invariance occur only at the one-loop level. Multi-loop diagrams can be regularized in a gauge covariant way by Pauli–Villars regularization of the internal bose lines these diagrams necessarily contain.

When an anomaly occurs, it is impossible to define an amplitude to obey all physical principles. In this situation, different attempts at defining the amplitude may give different answers. For instance, consider the triangle diagram. It should obey Bose symmetry and current conservation in each of three external lines. There are two standard ways to define the triangle. One may insist on Bose symmetry in

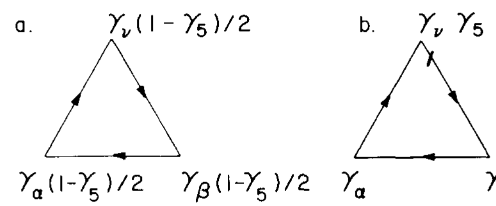

Fig. 5. Alternative forms of the fermion triangle.

each of the three lines and check for current conservation. Or one may insist on Bose symmetry and current conservation in two of the three lines and check for current conservation in the third one.

The difference is exemplified by the two diagrams of fig. 5. In fig. 5a there is a factor of $\frac{1}{2}\gamma_\mu(1-\gamma_5)$ at each vertex. In fig. 5b there are two insertions of γ_μ and one insertion of $\gamma_\nu\gamma_5$. Formally, for massless fermions, since $(\gamma_5)^2=1$ and γ_5 anticommutes with gamma matrices, fig. 5b is equal to just -2 times the parity-violating part of fig. 5a.

Suppose, though, that one defines these diagrams by Pauli–Villars regularization – by subtracting the contribution of a massive regulator field. Then figs. 5a and 5b are not equivalent in the presence of the regulator, and because of the anomaly they do not become equivalent even as the regulator mass goes to infinity. Fig. 5a, as the regulator mass goes to infinity, obeys Bose symmetry but violates current conservation in each channel, while fig. 5b respects Bose symmetry and current conservation in the vector channels but violates them in the axial vector channel.

If the amplitude of 5a obeyed current conservation its parity-violating part would serve as an acceptable definition of the amplitude of fig. 5b. If fig. 5b were conserved in the axial vector channel, then by Bose symmetrizing it one could get an acceptable definition of the amplitude of fig. 5a.

This observation is indispensable because in fact (at least if one uses Pauli–Villars regularization) it is much easier to calculate the amplitude of fig. 5b. When we calculate gravitational anomalies, the simplification from considering a diagram of type 5b will be essential.

As we have discussed, the existence of anomalies depends upon the fact that the one-loop diagrams cannot be regulated in a way that preserves chiral symmetry. Let us therefore examine this point briefly. To regularize the lagrangian

$$\mathscr{L}=\bar{\psi}i\not{D}\left(\frac{1-\gamma_5}{2}\right)\psi$$

by adding, say,

$$\frac{1}{\Lambda}\bar{\psi}D_\mu D^\mu\left(\frac{1-\gamma_5}{2}\right)\psi,$$

(with Λ as a cutoff parameter) is not useful because the additional term violates chiral symmetry explicitly. However, one could add higher-dimension terms that preserve chiral symmetry; for instance one could consider the chirally invariant lagrangian

$$\mathscr{L}' = \bar{\psi} i \not{D}\left(1 - \frac{1}{\Lambda^2} D_\mu D^\mu\right)\left(\frac{1-\gamma_5}{2}\right)\psi .$$

The reason that this regularization fails to eliminate anomalies is slightly subtle. Although $\mathscr{L}'$ conserves chiral symmetry, the naive current

$$J^a_\mu = \bar{\psi}\gamma_\mu \lambda^a\left(\frac{1-\gamma_5}{2}\right)\psi$$

is not conserved in the theory described by $\mathscr{L}'$. Rather one must find the appropriate conserved current by applying Noether's theorem to $\mathscr{L}'$. It includes additional pieces such as

$$\Delta J^a_\mu = \bar{\psi} i \gamma^\mu \lambda^a\left(-\frac{1}{\Lambda^2} D_\alpha D^\alpha\right)\psi .$$

In one-loop diagrams with gluons (or gravitons) coupled to the conserved currents derived from $\mathscr{L}'$, there are extra factors of momentum in the vertices which just cancel the improvement of the propagators. Therefore, the passage from $\mathscr{L}$ to $\mathscr{L}'$ as a regularization does not eliminate anomalies.

More generally, we can show that under broad assumptions the one-loop anomalies depend only on the quantum numbers of the elementary fields, and not on the specific lagrangian chosen. Let A and B be two appropriate differential operators; in the spin-$\frac{1}{2}$ case we may take $A = i\not{D}$ and $B = i\not{D}(1 - D_\mu D^\mu/\Lambda^2)$, for example. Assume that A and B conserve parity. Let us prove that the two theories with lagrangians

$$\mathscr{L}_1 = \bar{\psi} A\left(\frac{1-\gamma_5}{2}\right)\psi , \qquad \mathscr{L}_2 = \bar{\psi} B\left(\frac{1-\gamma_5}{2}\right)\psi$$

have the same one-loop anomalies. It is equivalent to prove that the theory with lagrangian

$$\mathscr{L} = \bar{\psi} A\left(\frac{1-\gamma_5}{2}\right)\psi + \bar{\psi} B\left(\frac{1+\gamma_5}{2}\right)\psi ,$$

is free of anomalies. But

$$\mathscr{L} = \bar{\psi}\tfrac{1}{2}(A+B)\psi - \bar{\psi}\tfrac{1}{2}(A-B)\gamma_5\psi ,$$

and a suitable regularization is simply to pass from $\mathscr{L}$ to

$$\mathscr{L}' = \bar{\psi}\tfrac{1}{2}(A+B)(1 + (-D_\mu D^\mu/\Lambda^2)^n)\psi - \bar{\psi}\tfrac{1}{2}(A-B)\gamma_5\psi ,$$

for some suitable integer n. This regularization improves the propagators in Feynman diagrams and it does not add extra powers of momentum in *parity-violating* vertices. Since all parity-violating amplitudes computed from $\mathscr{L}'$ are highly convergent, $\mathscr{L}'$ is free of anomalies, proving that the original theories with lagrangian

$$\bar{\psi}\!\!\!/A\left(\frac{1-\gamma_5}{2}\right)\psi, \qquad \text{or} \qquad \bar{\psi}\!\!\!/B\left(\frac{1-\gamma_5}{2}\right)\psi$$

had exactly the same anomalies. This observation will be quite useful when we calculate the gravitational anomaly of a spin-$\frac{3}{2}$ field in sect. 7.

3. Gravitational anomalies in four dimensions

Before attempting a systematic discussion of gravitational anomalies in n dimensions, let us discuss some implications of known facts in four dimensions.

Soon after the discovery of the standard triangle anomaly, it was pointed out that the fermion triangle with one axial current and two energy momentum tensors (fig. 3) has a similar anomaly [5]. This anomaly has been much discussed in connection with gravitational instantons. The axial vector current J^5_μ of massless fermions is not conserved, but obeys

$$D_\mu J^5_\mu = -\frac{1}{384\pi^2} R\tilde{R}\,, \tag{6}$$

where $R\tilde{R} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}R_{\mu\nu\sigma\tau}R_{\alpha\beta}{}^{\sigma\tau}$, $R_{\mu\nu\sigma\tau}$ being the usual Riemann curvature tensor. One of the results of our discussion in sect. 6 will be a relatively simple way to obtain eq. (6) from Feynman diagrams.

Violation of a global chirality conservation law in the presence of gravity is only one aspect of eq. (6). Another consequence arises if the anomalous axial current J^5_μ is coupled to a gauge field A_μ. In this case eq. (6) represents a breakdown of gauge invariance. It corresponds to an effective action Γ that is not gauge invariant but changes under the transformation $A_\mu \to A_\mu - \partial_\mu \varepsilon$ by an amount

$$\delta\Gamma = -\frac{1}{384\pi^2}\int \mathrm{d}^4x\,\sqrt{g}\,\varepsilon(x) R\tilde{R}(x)\,. \tag{7}$$

Actually, the anomaly takes this form if one defines the triangle diagram in a way that maintains general covariance and sacrifices current conservation. One may instead define a triangle amplitude that obeys current conservation and violates general covariance. This can be done as follows. The topological density $R\tilde{R}$ can be written as a total divergence $R\tilde{R} = D_\mu K^\mu$ where K^μ is a functional of the metric that is not generally covariant. One can replace the amplitude Γ by $\Gamma' = \Gamma + (1/384\pi^2)\int \mathrm{d}^4x\sqrt{g}A_\mu K^\mu$ which differs from Γ' by a local functional of the fields. One may readily see that Γ' is invariant under the gauge transformation $A_\mu \to A_\mu - \partial_\mu\varepsilon$. However, Γ' is not generally covariant because K^μ is not. It is not possible to define the triangle in a way that respects both gauge invariance and general covariance.

Therefore in four dimensions gauge theories cannot be consistently coupled to gravity unless the triangle anomaly of fig. 3 cancels when summing over all of the elementary Fermi fields.

Precisely this problem can potentially arise in the standard $SU(2) \times U(1)$ model of weak interactions. Let us take A_μ to be the U(1) gauge field that is coupled to hypercharge. Let Y be the hypercharge operator regarded as a matrix acting on the left-handed Fermi fields. The triangle diagram with one external hypercharge gauge boson and two external gravitons is proportional to Tr Y. A necessary condition for consistency of the $SU(2) \times U(1)$ model coupled to gravity is therefore

$$\operatorname{Tr} Y = 0 . \tag{8}$$

If this condition does not hold, then either gauge invariance or general covariance is lost at the one-loop level.

In fact, eq. (8) does hold in nature for the fermions of each observed generation. However, this condition is usually interpreted as evidence for grand unification; in grand unified theories, the hypercharge operator is a generator of a simple group and must be traceless. We see that since gravity does exist in nature, eq. (8) is needed for simple mathematical consistency.

One might be sceptical of this claim on the grounds that general relativity is unrenormalizable. How do we know that eq. (8) for cancellation of the anomaly will not be modified by whatever cures the short distance behavior of quantum gravity?

The point is that it is possible to understand anomalies strictly on the basis of *long wavelength* physics. For instance, consider trying to study the low-energy limit of the gauge boson-graviton-graviton coupling on the basis of unitarity. (See Coleman and Grossman, and Frishman et al., ref. [4].) By imposing unitarity in each channel, one could reconstruct the amplitude from the possible zero mass intermediate states, up to a term that is analytic in the momenta at $p = q = r = 0$. But the whole idea of the anomaly is that one cannot eliminate it by adding to the amplitude a term analytic in the momenta at zero momentum. Otherwise, on dimensional grounds, it would be a suitable polynomial of dimension four that could compensate for the anomaly, and one would simply add this polynomial to obtain an anomaly-free triangle.

Theories free of the usual triangle anomalies but with Tr $Y \neq 0$ are easily constructed. For instance, one may add to the standard model left-handed $SU(3) \times SU(2)$ singlets of hypercharges y_i. If $\sum y_i^3 = 0$, but $\sum y_i \neq 0$, this preserves the cancellation of the usual anomalies but introduces an anomaly in the coupling to gravity.

What specific consequences would this have? The usual electric charge operator is $Q = T_3 + \frac{1}{2} Y$. Since T_3 is traceless in any representation of SU(2), if Tr $Y \neq 0$ then Tr $Q \neq 0$. Such a world would therefore have massless electrically charged fermions

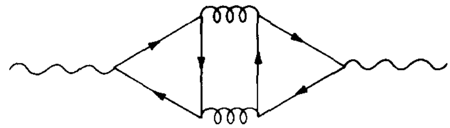

Fig. 6. A two-loop diagram that gives the photon a mass if Tr $Y \neq 0$. External wavy lines are photons; internal loopy lines are gravitons.

with parity-violating electromagnetic couplings. The anomaly would show up in diagrams with external photons and, by analogy with a similar discussion in the case of gauge theories [3], the photon would get a mass from a three-loop diagram with two internal gravitons (fig. 6). Despite the smallness of the gravitational constant, this photon mass is not necessarily negligible. It would be of order

$$m_\gamma^2 = \alpha G_N^2 \Lambda^6 , \tag{9}$$

where m_γ is the photon mass, G_N is Newton's constant, and Λ is an ultraviolet cut-off needed to make sense out of the divergent diagram. For $\Lambda = 500$ TeV, this gives a photon Compton wavelength of about 10^4 km, roughly the observational lower bound*. Thus the gravitational interactions would need to be cut off at rather "low" energies.

The anomaly of fig. 3 has one other interesting application. 't Hooft pointed out some years ago [4] that anomalies can serve as a restriction on the quantum numbers of composite massless particles. Lt J_μ be a conserved current in some quantum field theory, and assume that the corresponding conservation law is not spontaneously broken. Then the $\langle J_\mu J_\alpha J_\beta \rangle$ anomaly computed in terms of the elementary quanta must be precisely equal to the same anomaly computed in terms of the massless particles of the exact physical spectrum. Just the same condition must hold for the $\langle J_\mu T_{\alpha\beta} T_{\sigma\nu} \rangle$ anomaly. Therefore, the trace of any conserved charge Y evaluated among the elementary left-handed fermions must equal the same trace evaluated in the physical spectrum – unless the conservation of Y is spontaneously broken. This requirement should be imposed as a constraint in preon models.

4. Purely gravitational anomalies

In sect. 3, we discussed some implications of the anomaly that arises when gauge fields are coupled to gravity. It is natural to ask whether anomalies occur in theories with gravitational couplings only.

As we have discussed, the classical results about triangle anomalies concern the question of whether the fermion effective action is gauge invariant. Let us ask the analogous question for gravity. Let Γ be the one-loop effective action for matter

* We have assumed here that the triangle is defined in a way that preserves general coordinate invariance and sacrifices gauge invariance. Otherwise, the graviton would gain a mass.

fields propagating in a gravitational field. Γ is, of course, a functional of the metric tensor. Is Γ generally covariant? Or could it be that in some theories of matter fields coupled to gravity general covariance is violated at the one-loop level by anomalies?

Under the infinitesimal coordinate transformation $x^\mu \to x^\mu + \varepsilon^\mu$, the variation in the metric tensor is $\delta g_{\mu\nu} = -D_\mu \varepsilon_\nu - D_\nu \varepsilon_\mu$. The variation of Γ is

$$\delta\Gamma = -\int \mathrm{d}^4x \sqrt{g}\, \delta\Gamma/\delta g_{\mu\nu}(D_\mu \varepsilon_\nu + D_\nu \varepsilon_\mu).$$

But $\delta\Gamma/\delta g_{\mu\nu}$ is $\frac{1}{2}\langle T_{\mu\nu}\rangle$, where $\langle T_{\mu\nu}\rangle$ is the expectation value of the energy-momentum tensor of the matter fields. Integrating by parts and using the symmetry of $T_{\mu\nu}$, we have then that $\delta\Gamma = \int \mathrm{d}^4x \sqrt{g}\, \varepsilon_\nu D_\mu \langle T^{\mu\nu}\rangle$. So the question of whether the effective action is invariant under infinitesimal general coordinate transformations is equivalent to the question of whether the induced energy-momentum tensor of the matter fields is conserved. This is just analogous to the fact that in the case of gauge fields coupled to charged fermions, gauge invariance of the effective action is equivalent to conservation of the induced current.

Now, by reasoning analogous to the discussion in sect. 2, the real part of Γ is always generally covariant. But we will see that the imaginary part of Γ can suffer from anomalies.

But under what circumstances does Γ have an imaginary part? In euclidean space of n dimensions, the holonomy group of a riemannian manifold is $\mathrm{O}(n)$ (or a subgroup thereof). Consider matter fields in some representation Q of $\mathrm{O}(n)$. Their coupling to gravity involves the metric and connection – which are real – and the $\mathrm{O}(n)$ matrices in the Q representation, which may be complex. Only in case Q in a complex representation does Γ have an imaginary part. But $\mathrm{O}(n)$ has complex representations only if $n = 4k+2$ for some integer k, so it is only in $4k+2$ dimensions that the effective action may violate general covariance. In particular, for matter fields coupled to gravity in four dimensions, the one-loop action always respects general covariance.

Which complex representations of $\mathrm{O}(n)$ are relevant? For fermions, we may consider spin-$\frac{1}{2}$ fields of definite chirality or spin-$\frac{3}{2}$ fields of definite chirality. As we will see, each of these fields gives rise to one-loop anomalies. In addition, it seems that there is one type of Bose field that suffers from an anomaly. The simplest complex bosonic representation of $\mathrm{O}(4k+2)$ is an antisymmetric tensor $F_{\mu_1\cdots\mu_{2k+1}}$ with $2k+1$ indices that obeys a duality condition $F_{\mu_1\cdots\mu_{2k+1}} = \pm i/(2k+1)! \times \varepsilon_{\mu_1\cdots\mu_{2k+1}\nu_1\cdots\nu_{2k+1}} F^{\nu_1\cdots\nu_{k+1}}$. Certain very interesting theories [11, 17] contain an antisymmetric tensor field $A_{\mu_1\cdots\mu_{2k}}$ of $2k$ indices whose curl is constrained to obey such a condition. We will see that also for such a field, there is a one-loop anomaly. Most of the rest of this paper will consist of a detailed evaluation of anomalies for the spin-$\frac{1}{2}$, spin-$\frac{3}{2}$, and antisymmetric tensor fields.

First, however, let us discuss in more detail what is special about $4k+2$ dimensions. (See also ref. [18], where many of the points that follow have been made.)

In an odd number of dimensions the Lorentz group $O(1, n-1)$ has only one type of spinor representation. Its couplings to gravity conserve parity, leading to a real effective action that is free of perturbative anomalies.

In an even number of dimensions, the group $O(1, n-1)$ has two spinor representations related to each other by parity. One might hope to make a theory with parity-violating gravitational couplings by including fermions of one chirality only. This is only possible in $4k+2$ dimensions, for the following reason.

Let $\gamma_0, \gamma_1, \ldots \gamma_{n-1}$ be the Dirac gamma matrices, obeying $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$. (Our signature is $(+---\cdots-)$.) The operator that distinguishes the two spinor representations is $\gamma_5 = \gamma_0\gamma_1 \ldots \gamma_{n-1}$. It commutes with all generators of $O(1, n-1)$.

Now in $4k$ dimensions, one may readily see that $(\gamma_5)^2 = -1$. Hence the eigenvalues of γ_5 are $\pm i$, and are exchanged by complex conjugation. This means that in $4k$ dimensions the CPT operation reverses the chirality of fermions. Acting on fermions of $\gamma_5 = +i$, CPT gives fermions of $\gamma_5 = -i$, and vice versa. Hence in $4k$ dimensions, CPT conservation requires the existence of an equal number of fermions of positive and negative chirality. The couplings of fermions to gravity in $4k$ dimensions conserve parity (or violate parity only by irrelevant non-minimal terms) and the effective action is real.

In $4k+2$ dimensions, the story is different. In this case $\gamma_5^2 = +1$, so γ_5 has eigenvalues ± 1. A CPT transformation maps particles of given helicity into particles of the same helicity. In this case, it is possible to consider a theory in which the gravitational couplings are chirally asymmetric. One may have more fermion multiplets of one chirality than of the other. For such theories, the gravitational couplings violate parity, the euclidean space effective action is complex and, as we will see, there are anomalies.

One more side to this story deserves a mention. Specializing to fields of spin $\frac{1}{2}$, in four dimensions the basic object is the four-component Majorana field. General covariance permits this field to have a mass, and hence its couplings to gravity can be regularized in a generally covariant way by the Pauli–Villars method. This being so, the effective action to which it gives rise is generally covariant.

However, in $4k+2$ dimensions the basic object is the fermion field of definite chirality. General covariance (or even global Lorentz invariance) forbids such a field to have a bare mass; a fermion mass term in $4k+2$ dimensions is not allowed for Fermi fields all of the same chirality. Because the mass is forbidden, Pauli–Villars regularization cannot be performed, and anomalies may occur.

Our discussion so far has concerned the question of whether the effective action is invariant under *infinitesimal* general coordinate transformations. If so, one must still address the question of whether the effective action is invariant under *non-infinitesimal* general coordinate transformations – transformations that cannot be reached by exponentiating an infinitesimal transformation. This question which is analogous to certain considerations in gauge theories [16], will be our subject in sect. 10. Here let us simply note a few kinematical facts that are relevant.

In dimensions other than $4k+2$, the basic Majorana spinor representation* of $O(1, n-1)$ is either real or pseudoreal; that is, it admits a second-order invariant tensor that is either symmetric or antisymmetric. For instance, in four dimensions the four-component Majorana spinor representation admits an antisymmetric invariant tensor $\varepsilon_{\alpha\beta}$. Corresponding to this the mass term is $i\varepsilon_{\alpha\beta}\psi^{\alpha}\psi^{\beta}$ is possible (it is usually written $\bar{\psi}\psi$).

In eight or nine dimensions, or more generally in $8k$ or $8k+1$ dimensions, the situation is different. In $8k$ or $8k+1$ dimensions, the Majorana spinor representation admits only a *symmetric* invariant $c_{\alpha\beta}$. One cannot in $8k$ or $8k+1$ dimensions write a mass term for a single Majorana fermion, since $c_{\alpha\beta}\psi^{\alpha}\psi^{\beta}=0$ by Fermi statistics.

Given several Majorana fermions $\psi^{\alpha}{}_i$, $i=1\cdots k$, one can write the mass term $c_{\alpha\beta}\psi^{\alpha}{}_i\psi^{\beta}{}_jM_{ij}$ where M_{ij} is an antisymmetric mass matrix. If k is even, all fermions can obtain mass this way, but if k is odd, the antisymmetric matrix M necessarily has a zero eigenvalue. This means that with an odd number of Majorana spinors in $8k$ or $8k+1$ dimensions Pauli–Villars regularization is not possible – the regulator field would always have had an even number of components. This suggests that non-perturbatively there may be difficulties with an odd number of Majorana fermions in $8k$ or $8k+1$ dimensions, and we will see in sect. 10 that this is the case. As in the case of the Z_2 anomaly in gauge theories, the difficulty has to do with the sign of the fermion integral.

5. The spin-$\frac{1}{2}$ anomaly in two dimensions

In this section we will illustrate the preceding discussion with a detailed calculation of the gravitational anomaly for a spin-$\frac{1}{2}$ Weyl fermion in two dimensions. We will carry out this discussion with a Minkowski signature. The advantage of two dimensions is that because of the simplicity of the relevant diagram we can calculate explicitly the whole relevant amplitude and then study its behavior under coordinate transformations. In our subsequent study of loop diagrams in more than two dimensions, we will only be able to study the anomalous behavior of the loop diagrams.

In two dimensions the Dirac lagrangian for a fermion in a gravitational field can be written

$$\mathscr{L}=\int \mathrm{d}^2x \det e\, \mathrm{e}^{\nu a}(\tfrac{1}{2}\bar{\psi}i\gamma_a\overleftrightarrow{\partial}_{\nu}\psi)\,. \tag{10}$$

Here $e_{\nu a}$ is the tetrad; the spin connection drops out of the Dirac lagrangian in two dimensions because of Fermi statistics. We will study the propagation of fermions

* By the Majorana spinor representation we mean the minimum-dimensional representation of the Clifford algebra by gamma matrices that are all real or all imaginary. In $8k$ or $8k+1$ dimensions they are real; for other values of the dimension they are imaginary.

in the weak gravitational field $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$. We will investigate the behavior of the effective action under coordinate transformations, not local Lorentz transformations, so it will be adequate to make a simple gauge choice for the tetrad: $e_{\mu a}=\eta_{\mu a}+\frac{1}{2}h_{\mu a}$. At the linearized level, the interaction lagrangian is simply $\Delta\mathscr{L}=-\frac{1}{2}h^{\mu\nu}T_{\mu\nu}$ where $T_{\mu\nu}=\frac{1}{4}i\bar\psi(\gamma_\mu\overleftrightarrow{\partial}_\nu+\gamma_\nu\overleftrightarrow{\partial}_\mu)\psi$ is the fermion energy-momentum tensor.

We wish to calculate the effective action for fermions that obey a Weyl condition $\gamma_5\psi=-\psi$, where $\gamma^5=\gamma^0\gamma^1$. (We will discuss complex fermions that obey this condition, although in two dimensions one could impose a Majorana–Weyl condition; this involves dividing the effective action by two.) It is convenient to introduce light-cone coordinates $x^\pm=\sqrt{\frac{1}{2}}(x^0\pm x^1)$. The corresponding gamma matrices $\gamma^\pm=\sqrt{\frac{1}{2}}(\gamma^0\pm\gamma^1)$ obey $(\gamma^+)^2=(\gamma^-)^2=0$, $\gamma^+\gamma^-+\gamma^-\gamma^+=2$. Indices are raised and lowered as follows: $V^+=V_-$, $V^-=V_+$, $V^\mu W_\mu=V^+W_++V^-W_-=V_+W_-+V_-W_+$.

A fermion obeying $\gamma_5\psi=-\psi$ also obeys $0=\gamma^-\psi=\gamma_+\psi$. For this reason, the free equation of motion $0=(\gamma_+\partial_-+\gamma_-\partial_+)\psi$ reduces to $0=\partial_-\psi=\sqrt{\frac{1}{2}}(\partial/\partial t-\partial/\partial x)\psi$. So in two dimensions a fermion of negative chirality is simply an object that travels constantly in the $+x$ direction at the speed of light. Such an object, of course, cannot have a mass – it cannot be brought to rest – and this is ultimately why anomalies are possible, as we have previously discussed.

With $\gamma_-\psi=\partial_-\psi=0$, the only non-vanishing component of the energy-momentum tensor is $T_{++}=\frac{1}{2}i\bar\psi\gamma_+\overleftrightarrow{\partial}_+\psi$, and the linearized interaction of fermions with the gravitational field is $\Delta\mathscr{L}=-h_{--}\frac{1}{4}i\bar\psi\gamma_+\overleftrightarrow{\partial}_+\psi$. We will study the effective action to second order in the metric perturbation h, by studying the two-point function

$$U(p)=\int d^2x\, e^{ip\cdot x}\langle\Omega|T(T_{++}(x)T_{++}(0))|\Omega\rangle\,. \tag{11}$$

Now, it is possible to see without any computation that there must be an anomaly. The naive conservation law for T_{++} is $\partial_-T_{++}=0$; it leads to the naive Ward identity $p_-U=0$*. If true, this would imply $U=0$ for all non-zero p_-, and hence (by analyticity) for all p_-. But U, as the two-point function of the hermitian operator T_{++}, cannot vanish. So there must be an anomaly.

In fact, the anomaly is easily computed, using tricks introduced in [19]. After performing the Dirac algebra in fig. 7, one finds

$$\begin{aligned}U(p)&=-\tfrac{1}{4}\int\frac{dk_+dk_-}{(2\pi)^2}(2k+p)_+{}^2\frac{k_+}{k_+k_-+i\varepsilon}\,\frac{(k+p)_+}{(k+p)_+(k+p)_-+i\varepsilon}\\&=-\tfrac{1}{4}\int\frac{dk_+\,dk_-}{(2\pi)^2}(2k+p)_+{}^2\frac{1}{k_-+i\varepsilon/k_+}\,\frac{1}{(k+p)_-+i\varepsilon/(k+p)_+}\,.\end{aligned} \tag{12}$$

* Naively there is no equal time commutator term in this Ward identity. If one looks at (11) as a two-point function in flat space, the anomaly we will find can be regarded as an anomalous commutation relation $[T_{++}(x),T_{++}(y)]=(i/48\pi)\delta'''(x-y)+$tree level terms. It is closely related to the anomaly in the Virasoro algebra in string theories. But we will see that upon coupling T_{++} to the gravitational field, the anomaly is a breakdown of general covariance.

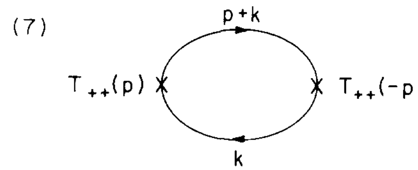

Fig. 7. The gravitational anomaly in two dimensions.

One now performs first the k_- integral by contour integration. It vanishes unless the poles at $k_- = -i\varepsilon/k_+$ and $k_- = -p_- - i\varepsilon/(k+p)_+$ are on opposite sides of the real axis. So if, say, $p_+ > 0$, the k_+ integral can be restricted to $0 > k_+ > -p_+$. We have then

$$U(p) = \frac{i}{8\pi}\int_{-p_+}^{0} \mathrm{d}k_+ \frac{(2k+p_+)^2}{p_-}$$
$$= \frac{i}{24\pi}\frac{p_+{}^3}{p_-}. \tag{13}$$

So $U(p) \neq 0$. What is more, as expected the anomaly is finite and is a polynomial in the momenta:

$$p_- U(p) = \frac{i}{24\pi} p_+{}^3 . \tag{14}$$

Now, let us discuss the question of the covariance of the effective action. If we couple $-\frac{1}{2}h_{--}$ at each vertex in fig. 7, then this diagram represents $i\mathscr{L}_{\mathrm{eff}}(h_{--})$, $\mathscr{L}_{\mathrm{eff}}$ being the effective action for the gravitational field. Remembering to include a factor of $\frac{1}{2}$ for Bose statistics, (13) corresponds to the effective action

$$\mathscr{L}_{\mathrm{eff}}(h_{\mu\nu}) = -\frac{1}{192\pi}\int \mathrm{d}^2 p \frac{p_+{}^3}{p_-} h_{--}(p) h_{--}(-p) , \tag{15}$$

where, of course, $h_{--}(p)$ is the Fourier transform of the metric perturbation. We wish to discuss the behavior of (15) under coordinate transformations.

Under an infinitesimal general coordinate transformation $\delta x^\mu = \varepsilon^\mu$, $h_{\mu\nu}$ transforms as $\delta h_{\mu\nu}(x) = -\partial_\mu \varepsilon_\nu(x) - \partial_\nu \varepsilon_\mu(x)$. A coordinate transformation therefore corresponds in momentum space to

$$\delta h_{++}(p) = -2ip_+\varepsilon_+ ,$$
$$\delta h_{+-}(p) = -ip_-\varepsilon_+ - ip_+\varepsilon_- ,$$
$$\delta h_{--}(p) = -2ip_-\varepsilon_- . \tag{16}$$

Now, it is obvious that (15) is not invariant under this transformation; but that does not quite prove the existence of an anomaly. We must try to use our freedom of adding to the effective action a local functional of the fields so as to obtain a gauge invariant effective action. So we try $\mathscr{L}_{\text{eff}} \to \mathscr{L}_{\text{eff}} + \Delta\mathscr{L}$, where

$$\Delta\mathscr{L} = \int d^2p[Ap_+{}^2h_{--}(p)h_{+-}(-p) + Bp_+p_-h_{+-}(p)h_{+-}(-p) + Cp_+p_-h_{++}(p)h_{--}(-p) + Dp_-{}^2h_{++}(p)h_{+-}(-p)], \quad (17)$$

this being the most general Lorentz invariant polynomial of appropriate dimension. It is easy to see that regardless of the choice of A, B, C, and D the modified action $\mathscr{L}_{\text{eff}} + \Delta\mathscr{L}$ is not invariant under (16). This concludes the demonstration that the effective action for a Weyl fermion in two dimensions is not generally covariant.

To complete the picture, suppose we consider a massless Dirac fermion in $1+1$ dimensions, with both chiralities present. The effective action obtained from Feynman diagrams can be found by adding (15) to its parity conjugate:

$$\mathscr{L}_{\text{eff}}^{\text{Dirac}} = -\frac{1}{192\pi}\int d^2p\left(\frac{p_+{}^3}{p_-}h_{--}(p)h_{--}(-p) + \frac{p_-{}^3}{p_+}h_{++}(p)h_{++}(-p)\right). \quad (18)$$

Again, this is not invariant under coordinate transformations. Now, however, we can add a local counterterm to form an action

$$\bar{\mathscr{L}} = -\frac{1}{192\pi}\int d^2p\left(\frac{p_+{}^3}{p_-}h_{--}(p)h_{--}(-p) + \frac{p_-{}^3}{p_+}h_{++}(p)h_{++}(-p) + 2p_+p_-h_{++}(p)h_{--}(-p) - 4p_+{}^2h_{--}(p)h_{+-}(-p) - 4p_-{}^2h_{++}(p)h_{+-}(-p) + 4p_+p_-h_{+-}(p)h_{+-}(-p)\right), \quad (19)$$

which is easily seen to be invariant under coordinate transformations. It can be written more succinctly as

$$\bar{\mathscr{L}} = -\frac{1}{192\pi}\int d^2p\frac{R(p)R(-p)}{p_+p_-}, \quad (20)$$

where $R = p_+{}^2h_{--} + p_-{}^2h_{++} - 2p_+p_-h_{+-}$ is the linearized form of the curvature scalar.

As a by-product of this investigation we can extract a formula for the well-known trace anomaly. Classically, the energy momentum tensor for the massless Dirac field is traceless; it obeys $T_{+-} = 0$. As a consequence, at the linearized level h_{+-} does not couple to fermions and should not appear in the linearized effective action. We see, indeed, that h_{+-} is absent in the effective action (18) obtained from diagrams, but a dependence on h_{+-} is needed in (19) for general covariance.

The trace of the induced energy momentum tensor for a Dirac field in curved space is

$$\langle 2T_{+-}(p)\rangle = -2\frac{\delta\bar{\mathscr{L}}}{\delta h_{+-}(-p)} = -\frac{1}{24\pi}R(p);$$

this is the equation of the trace anomaly.

The evaluation of Feynman diagrams in more than two dimensions is considerably more difficult. It is useful to note that a simple argument ensures that if there is an anomaly in two dimensions, there is also an anomaly in $4k+2$ dimensions, for any k.

Consider a theory of spin-$\frac{1}{2}$ Weyl fermions in $4k+2$ dimensions. Suppose that the $4k+2$ dimensional space is of the form $\mathrm{M}^2 \times \mathrm{B}$ where M^2 is an asymptotically Minkowskian two-dimensional world and B is a compact space of $4k$ dimensions on which the Dirac field has a non-zero index. Massless fermions in the effective two-dimensional world are zero modes of the Dirac operator on B. The usual relation $\gamma_1\gamma_2\cdots\gamma_{4k+2} = (\gamma_1\gamma_2)\cdot(\gamma_3\gamma_4\cdots\gamma_{4k+2})$ shows that for chiral fermions in $4k+2$ dimensions (say $\gamma_1\gamma_2\cdots\gamma_{4k+2}=+1$) the two-dimensional chirality (eigenvalue of $\gamma_1\gamma_2$) equals the $4k$-dimensional chirality (eigenvalue of $\gamma_3\gamma_4\cdots\gamma_{4k+2}$). Hence if there is a non-zero index of the Dirac operator on B, so that the zero modes on B have preferentially one chirality, then the chiral theory in $4k+2$ dimensions reduces macroscopically to a chiral theory in two dimensions.

An anomaly free theory in n dimensions always remains anomaly free after any process of compactification. After all, absence of anomalies means that the Dirac determinant in n dimensions is well defined and generally covariant; if this is so the Dirac determinant must remain generally covariant after any valid approximation, such as an approximate reduction to a two-dimensional determinant. Since the Weyl theory in $4k+2$ dimensions can reduce – in the manner just described – to a Weyl theory in two dimensions, which we know to have an anomaly, the theory of Weyl fermions in $4k+2$ dimensions must have an anomaly for any k.

The restriction to $4k+2$ dimensions emerges, in this context, because the index of the Dirac operator always vanishes except in $4k$ dimensions.

As we will see in detail in subsequent sections, the anomaly in $4k+2$ dimensions involves several tensor structures. Not all of them survive reduction to two dimensions, so the trick of dimensional reduction is not a full substitute for a computation in $4k+2$ dimensions. But it does show that there is an anomaly in $4k+2$ dimensions for any k.

6. The spin-$\frac{1}{2}$ anomaly in $4k+2$ dimensions

In this section we will perform a diagrammatic evaluation of the gravitational anomaly in $4k+2$ dimensions.

We consider a diagram with n external graviton lines. The amplitude will depend on the n momenta $p_\mu^{(i)}$, $i=1\cdots n$ of the external gravitons and on their n symmetric polarization tensors $\varepsilon_{\mu\nu}^{(i)}$, $i=1\cdots n$. The momenta are not independent but obey one constraint, $\sum_{i=1}^n p_\mu^{(i)}=0$.

By our general considerations, only the parity-violating amplitudes are anomalous. A parity-violating amplitude is necessarily proportional to the Levi–Civita symbol $\varepsilon_{\mu_1\mu_2\cdots\mu_{4k+2}}$, which must be contracted with the external momenta and polarization vectors. The epsilon symbol, being antisymmetric, can be contracted with at most one index from each symmetric tensor $\varepsilon_{\mu\nu}$, and with at most $(n-1)$ linearly independent momentum vectors p_μ. To saturate the epsilon symbol it must be, therefore, that $n+(n-1)\geqslant 4k+2$ or $n\geqslant 2k+2$. We see, then, that in $4k+2$ dimensions, diagrams with less than $2k+2$ external gravitons are free from anomalies. We will evaluate the anomaly in one-loop diagrams with precisely $2k+2$ external gravitons. Diagrams with more than $2k+2$ gravitons also have anomalies which probably can be determined in terms of the anomalous diagrams with $2k+2$ gravitons by consistency conditions of the Wess–Zumino type.

The lagrangian for a Weyl fermion in $4k+2$ dimensions is

$$S=\int \mathrm{d}x \det e\, \mathrm{e}^{\mu a}\tfrac{1}{2}\bar{\psi} i\gamma_a \overleftrightarrow{D}_\mu\left(\frac{1-\gamma_5}{2}\right)\psi . \tag{21}$$

The covariant derivative of a spinor is $D_\mu\psi=\partial_\mu\psi+\frac{1}{2}\omega_{\mu cd}\sigma^{cd}\psi$, where $\omega_\mu{}^{cd}$ is the spin connection and $\sigma^{cd}=\frac{1}{4}[\gamma^c,\gamma^d]$. So $S=S_1+S_2$, with

$$\begin{aligned}
S_1&=\tfrac{1}{2}\int \mathrm{d}x \det e\, e^{\mu a}\bar{\psi} i\gamma_a \overleftrightarrow{\partial}_\mu\left(\frac{1-\gamma_5}{2}\right)\psi ,\\
S_2&=\tfrac{1}{2}\int \mathrm{d}x \det e\, e^{\mu a}\omega_\mu^{cd}\, i\bar{\psi}\{\gamma_a,\tfrac{1}{2}\sigma_{cd}\}\left(\frac{1-\gamma_5}{2}\right)\psi ,\\
&=\tfrac{1}{4}\int \mathrm{d}x \det e\, e^{\mu a}\omega_\mu^{cd}\, i\bar{\psi}\Gamma_{acd}\left(\frac{1-\gamma_5}{2}\right)\psi ,
\end{aligned} \tag{22}$$

where Γ_{acd} is the antisymmetrized product of three gamma matrices, $\Gamma_{acd}=\frac{1}{6}(\gamma_a\gamma_c\gamma_d\pm\text{permutations})$. We will take ψ to be a complex spinor restricted only by the condition $\gamma_5\psi=-\psi$. In $8k+2$ dimensions (but not in $8k+6$ dimensions), it would be possible to restrict ψ by a Majorana condition, and in this case our subsequent formulae must be divided by two.

As in sect. 5 we study a metric $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$, and work in the gauge $e_{\mu a}=\eta_{\mu a}+\frac{1}{2}h_{\mu a}$. The restriction to this gauge means that we will study anomalies in general coordinate transformations but not in local Lorentz transformations (though these could be studied by similar methods).

Vertices originating in S_1 have one or more external gravitons. Vertices originating in S_2 have at least two external gravitons, because, although $\mathrm{e}^{\mu a}\omega_\mu^{cd}$ contains a piece linear in $h_{\mu\nu}$, this vanishes when antisymmetrized in a, c, and d.

Parity violation only appears because of the factor of $\frac{1}{2}(1-\gamma_5)$ in the lagrangian. Of course, a trace containing γ_5 will vanish unless at least $4k+2$ gamma matrices are present. Actually, as we will see, in contracting a graviton line with its momentum to test for conservation, we always lose two gamma matrices. Hence, only diagrams with at least $4k+4$ gamma matrices in the numerator will be anomalous.

Consider a one-loop diagram with n_1 vertices originating from S_1 and n_2 vertices originating from S_2. Such a diagram has n_1+n_2 internal propagators, each with one gamma matrix in the numerator. It also has one gamma matrix at each vertex originating in S_1 and three gamma matrices at each vertex originating in S_2. The number of gamma matrices in the numerator of such a diagram is therefore $(n_1 + n_2) + n_1 + 3n_2 = 2(n_1 + 2n_2)$. We need, therefore, $2(n_1 + 2n_2) \geq 4k + 4$ if the diagram is to be anomalous, so $n_1 + 2n_2 \geq 2k + 2$.

We can get an inequality that runs in the opposite direction by counting external graviton lines. We want to look at diagrams with $2k+2$ external gravitons. There will be at least one external graviton for each vertex originating in S_1, and at least two for each vertex originating in S_2 so $2k + 2 \geq n_1 + 2n_2$ for the diagrams of interest.

Combining these inequalities, we see that the anomalous diagrams with $2k+2$ gravitons have $n_1 + 2n_2 = 2k + 2$, with precisely one graviton line attached to each vertex that originates in S_1 and precisely two attached to each vertex that originates in S_2. The interaction lagrangian therefore simplifies drastically; we may take

$$\mathscr{L}_1 = -\tfrac{1}{4} i h^{\mu\nu} \bar{\psi} \gamma_\mu \overleftrightarrow{\partial}_\nu \left(\frac{1-\gamma_5}{2} \right) \psi ,$$

$$\mathscr{L}_2 = -\tfrac{1}{16} i (h_{\lambda\alpha} \overleftrightarrow{\partial}_\mu h_{\nu\alpha}) \bar{\psi} \Gamma^{\mu\lambda\nu} \left(\frac{1-\gamma_5}{2} \right) \psi . \tag{23}$$

The Feynman rules for these vertices are given in fig. 8.

Now we must discuss how we will regularize the one-loop diagram. We will use a procedure discussed at the end of sect. 2 and originally due to Adler. Although the one-loop diagrams have Bose symmetry in the external lines, the simplest method for extracting the anomaly does not preserve this symmetry. Instead of placing a factor $\frac{1}{2}(1-\gamma_5)$ at each vertex, we place such a factor at one vertex only. Then we introduce a Pauli–Villars regulator, subtracting from our diagram a similar diagram with a massive fermion of mass M (or we add and subtract suitable diagrams with regulator fields of suitable masses; but it is not necessary to do this explicitly). The amplitude constructed in this way has an anomaly only in the channel where $\frac{1}{2}(1-\gamma_5)$ is inserted. By Bose symmetrization, one can construct the anomaly that

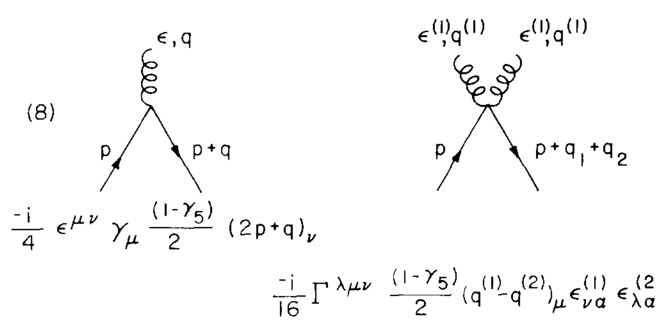

Fig. 8. The relevant interaction vertices for gravitons interacting with spin-$\frac{1}{2}$ fermions. The two vertices originate from $\mathscr{L}_1$ and $\mathscr{L}_2$, respectively.

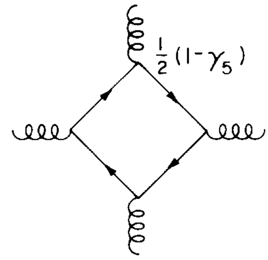

Fig. 9. An anomalous diagram with vertices originating from $\mathscr{L}_1$ only. Shown is the anomalous box diagram of six dimensions. A factor $\frac{1}{2}(1-\gamma_5)$ is inserted at one vertex only. From this amplitude is subtracted a like amplitude for a regulator fermion of mass M.

the Bose symmetric amplitude would have; it is equal, in each channel, to $1/(2k+2)$ times the anomaly we will calculate in one channel.

To indicate how the calculation goes we consider first a diagram (fig. 9) with vertices coming from $\mathscr{L}_1$ only. In the dangerous channel with an insertion of $\frac{1}{2}(1-\gamma_5)$ we take the graviton momentum to be p_μ and its polarization tensor to be $i(p_\mu\varepsilon_\nu + p_\nu\varepsilon_\mu)$; the amplitude with this polarization tensor should vanish because of invariance under the general coordinate transformation $x^\mu \to x^\mu + \varepsilon^\mu$. The other gravitons have momenta $p_\mu^{(i)}$, and to keep the algebra so simple as possible we assume a factorized form $\varepsilon_{\mu\nu}^{(i)} = \varepsilon_\mu^{(i)}\varepsilon_\nu^{(i)}$ for their polarization tensors; the general result can easily be reconstructed from this case.

As pointed out originally by Adler, in the regularized amplitude naive manipulations can be carried out. The anomaly appears only because for the massive regulator fermion, which we will call λ, the energy momentum tensor with insertion of $\frac{1}{2}(1-\gamma_5)$ is not conserved even formally. And it is only the regulator diagram that contributes to the anomaly.

In fact, if the regulator λ obeys $(i\not{D}-M)\lambda=0$, then $D_\nu(\bar\lambda\gamma_\mu\overleftrightarrow{D}_\nu\frac{1}{2}(1-\gamma_5)\lambda)=0$ while $D_\mu(\bar\lambda\gamma_\mu\overleftrightarrow{D}_\nu\frac{1}{2}(1-\gamma_5)\lambda)\neq 0$. Hence in the polarization tensor $i(p_\mu\varepsilon_\nu+p_\nu\varepsilon_\mu)$ of the dangerous channel, the term $p_\nu\varepsilon_\mu$ can be dropped; the other term causes trouble.

In fact, for a fermion of mass M, $D_\mu(\bar\lambda\gamma^\mu\overleftrightarrow{D}_\nu\frac{1}{2}(1-\gamma_5)\lambda)=-iM\bar\lambda\overleftrightarrow{D}_\nu\gamma_5\lambda$. Consequently, in the dangerous channel we may replace $i(p_\mu\varepsilon_\nu+p_\nu\varepsilon_\mu)\cdot(-\frac{1}{4}i\bar\lambda\gamma_\mu\overleftrightarrow{D}_\nu\frac{1}{2}(1-\gamma_5)\lambda)$ by $-\frac{1}{4}M\varepsilon^\nu\bar\lambda\overleftrightarrow{D}_\nu\gamma_5\lambda$. (Note the promised disappearance of a gamma matrix in this manipulation.)

In effect, then, fig. 9 becomes fig. 10, with $2k+1$ external gravitons and one insertion of $-\frac{1}{4}M\varepsilon^\nu\bar\lambda\overleftrightarrow{D}_\nu\gamma_5\lambda$.

As the amplitude of fig. 10 is somewhat complicated, let us proceed in stages. First we carry out the Dirac algebra, after putting the propagators in the usual rationalized form $i(\not{p}+M)/(p^2-M^2)$. The diagram contains exactly one factor of γ_5, at the anomalous vertex labeled Q. Remembering that the polarization vectors have been written $\varepsilon_{\mu\nu}^{(i)}=\varepsilon_\mu^{(i)}\varepsilon_\nu^{(i)}$, so that each non-anomalous vertex contains a factor

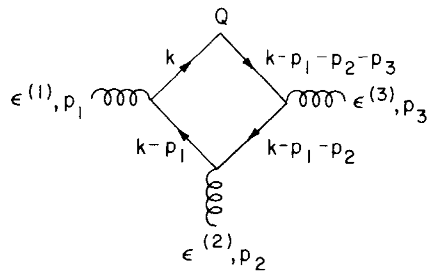

Fig. 10. After manipulations described in the text, the diagram of fig. 10 reduces to a one-loop diagram with one insertion of $Q=-\frac{1}{4}M\varepsilon^{\nu}\bar{\lambda}\overleftrightarrow{D}_{\nu}\gamma_5\lambda$ and several external gravitons. Fig. 10 is labeled more explicitly than fig. 9, to facilitate comparison with eq. (24).

of $\not{\varepsilon}^{(i)}$, the Dirac trace that must be performed is

$$\begin{aligned} A = \mathrm{Tr}\ \gamma_5(\not{k}+M)\not{\varepsilon}^{(1)}(\not{k}-\not{p}^{(1)}+M)\not{\varepsilon}^{(2)}(\not{k}-\not{p}^{(1)}-\not{p}^{(2)}+M) \\ \times \not{\varepsilon}^{(3)}\cdots\not{\varepsilon}^{(2k+1)}(\not{k}-\not{p}^{(1)}\cdots-p^{(2k+1)}+M)\,. \end{aligned} \tag{24}$$

The same trace was encountered in ref. [7]. There are at most $4k+3$ gamma matrices multiplying γ_5. A non-zero trace requires $4k+2$ gamma matrices, so we must pick out terms that are precisely linear in M. The various terms obtained by extracting M from different places nearly cancel each other. Bearing in mind that

$$\mathrm{Tr}\ \gamma_5\gamma_{\mu_1}\gamma_{\mu_2}\cdots\gamma_{\mu_{4k+2}} = -2^{2k+1}\varepsilon_{\mu_1\mu\cdots\mu_{4k+2}}\,,$$

the result is $A=2^{2k+1}MR(\varepsilon^{(i)},p^{(j)})$ where

$$R(\varepsilon^{(i)},p^{(j)}) = -\varepsilon_{\mu_1\mu_2\cdots\mu_{4k+2}}p^{(1)}_{\mu_1}\varepsilon^{(1)}_{\mu_2}p^{(2)}_{\mu_3}\varepsilon^{(2)}_{\mu_4}\cdots p^{(2k+1)}_{\mu_{4k+1}}\varepsilon^{(2k+1)}_{\mu_{4k+2}}\,. \tag{25}$$

The important point is that the kinematical factor R depends only on the external momenta and polarization vectors, and not on the loop momentum.

After eliminating the Dirac algebra in this way, the propagators are effectively $i/(p^2-M^2)$ – the propagator of a massive scalar field. At the ith vertex is a factor $-\frac{1}{4}i\varepsilon^{(i)}_{\mu}(p+p')^{\mu}$ where p and p' are the incoming and outgoing momenta of the scalar particles. At the anomalous vertex is a factor $-\frac{1}{4}i\varepsilon_{\mu}(p+p')^{\mu}$ where ε_{μ}, which we will henceforth call $\varepsilon^{(0)}_{\mu}$, is the parameter of an infinitesimal coordinate transformation.

These diagrams will appear formidable. But a little thought shows that the vertex factor $-\frac{1}{4}i\varepsilon^{(i)}_{\mu}(p+p')^{\mu}$ has a simple interpretation. It is the amplitude for the absorption by a charged scalar of charge $\frac{1}{4}$ of a photon of polarization $\varepsilon^{(i)}_{\mu}$ and momentum $(p'-p)^{\mu}$.

A charged scalar also has seagull vertices, of course, where two photons of polarization $\varepsilon^{(1)}$ and $\varepsilon^{(2)}$ are simultaneously absorbed, the amplitude being $2ie^2\varepsilon^{(1)}\varepsilon^{(2)}$. In our problem, however, there are gravitational seagull diagrams, coming from $\mathcal{L}_2$ of eq. (23). It is easy to see that after factoring out the Dirac algebra in the way just described, the gravitational seagulls become electromagnetic seagulls in the analogue problem. Putting the pieces together, the anomalous divergence of the amplitude with $2k+2$ gravitons is

$$I_{1/2}=2^{2k+1}iM^2R(\varepsilon^{(i)},p^{(j)})Z(\varepsilon^{(i)},p^{(j)})\,, \tag{26}$$

where Z is the amplitude for a charged scalar of mass M and charge $\frac{1}{4}$ interacting with $2k+2$ photons of momentum and polarization $p^{(j)}$ and $\varepsilon^{(i)}$, $i,j=0,\ldots,2k+1$.

In general, the evaluation of Z is quite formidable. However, we are interested in the limit as the regulator mass M goes to infinity. Z is gauge invariant and so must be constructed from the electromagnetic field strength $F_{\mu\nu}$ and its derivatives. Since Z is of order $(2k+2)$ in the field strengths, we see by dimensional analysis that in $4k+2$ dimensions, Z vanishes at least as $1/M^2$ for large M. Moreover, terms in Z involving derivatives of field strengths will vanish for large M faster than $1/M^2$ and are negligible. This means that we can regard Z as the amplitude for a scalar propagating in a *constant* electromagnetic field

$$F_{\mu\nu}=-i\sum_{j=0}^{2k+1}(p_\mu^{(j)}\varepsilon_\nu^{(j)}-p_\nu^{(j)}\varepsilon_\mu^{(j)})\,.$$

The amplitude Z for propagation in a constant field is to be evaluated and then expanded in powers of F to extract the term linear in each $p^{(i)}$ and $\varepsilon^{(j)}$.

A diagrammatic evaluation of Z would still be formidable, but the amplitude for propagation in a constant field can be evaluated by a method due to Schwinger [20]. It is convenient to perform the rest of the calculation in euclidean space. The effective action density is

$$\begin{aligned}Z&=\frac{1}{\mathrm{vol}}\ln\det(-D_\mu D^\mu+M^2)\\&=-\frac{1}{\mathrm{vol}}\int_0^\infty\frac{\mathrm{d}s}{s}\mathrm{Tr}\,\mathrm{e}^{s(D_\mu D^\mu)}\,\mathrm{e}^{-sM^2}\,.\end{aligned} \tag{27}$$

The electromagnetic field can be brought to canonical form

$$F_{\mu\nu}=2\begin{pmatrix} & x_1 & & & & \\ -x_1 & & & & & \\ & & & x_2 & & \\ & & -x_2 & & & \\ & & & \ddots & & x_{2k+1} \\ & & & & -x_{2k+1} & \end{pmatrix}, \tag{28}$$

with "eigenvalues" $2x_1,2x_2,\ldots,2x_{2k+1}$. (The factor of 2 is included for later convenience.)

For a particle of charge e interacting in two dimensions with a magnetic field of strength B, it is a classic result that [20]

$$\frac{1}{\mathrm{Vol}} \mathrm{Tr}\, \mathrm{e}^{s(D_\mu D^\mu)} = \frac{1}{4\pi} \frac{eB}{\sinh(eBs)} . \tag{29}$$

With the field brought to the canonical form (28) the "hamiltonian" $H = -D_\mu D^\mu$ in (27) is a sum of $2k+1$ commuting two-dimensional operators. The trace in (27) is therefore a product of two-dimensional traces, so we get

$$Z = -\int_0^\infty \frac{\mathrm{d}s}{s} \prod_{i=1}^{2k+1} \left(\frac{\frac{1}{2}x_i}{4\pi \sinh(x_i s)} \right) \exp(-sM^2) . \tag{30}$$

In (30) the factor in brackets is to be expanded to order $2k+2$ in the x_i; terms of lower or higher order are irrelevant. The term of order $(2k+2)$ in x_i is of order s, cancelling the $1/s$ singularity in (30). The s integral is then $\int_0^\infty \mathrm{d}s\, \mathrm{e}^{-sM^2}$. So (30) reduces to

$$Z = -\frac{1}{(4\pi)^{2k+1}} \frac{1}{M^2} \prod_{i=1}^{2k+1} \frac{\frac{1}{2}x_i}{\sinh \frac{1}{2}x_i} , \tag{31}$$

where it is understood that the right-hand side of (31) is to be expanded in powers of the x_i, with all terms dropped except terms of order $2k+2$.

Combining (26) and (31), we have finally for the anomaly due to a complex spin-$\frac{1}{2}$ Weyl fermion

$$I_{1/2} = -i \frac{1}{(2\pi)^{2k+1}} R(\varepsilon^{(i)}, p^{(j)}) \prod_{i=1}^{2k+1} \frac{\frac{1}{2}x_i}{\sinh \frac{1}{2}x_i} . \tag{32}$$

In interpreting this formula, it is to be understood that (32) is regarded as a function of $F_{\mu\nu}$ via eq. (28), and that $F_{\mu\nu}$ is to be expressed in terms of the $\varepsilon_\mu^{(i)}$ and $p_\nu^{(j)}$ by a formula given earlier. The product on the right-hand side of (32) is to be expanded, extracting terms linear in each $\varepsilon^{(i)}$ and $p^{(j)}$.

For $k=0$, (32) agrees with our earlier two-dimensional results (13) and (15). The comparison of (32) with those results is somewhat subtle because (15) was defined to obey Bose symmetry while (32) was computed with a regularization that violates Bose symmetry. The simplest way to make the comparison is to add to (13) suitable contact terms to construct a non-Bose-symmetric functional of the two graviton channels that is conserved in one channel. One then obtains (32) as the anomalous divergence in the other channel.

Eq. (32) is related to the formula for the Dirac index density in $4k+4$ dimensions, in agreement with remarks originally made by M.F. Atiyah (private communication). The linearized Riemann tensor of a graviton is

$$R_{\mu\nu\alpha\beta} = \tfrac{1}{2}(p_\mu p_\alpha \varepsilon_{\nu\beta} + p_\nu p_\beta \varepsilon_{\mu\alpha} - p_\mu p_\beta \varepsilon_{\nu\alpha} - p_\nu p_\alpha \varepsilon_{\mu\beta}) .$$

If $\varepsilon_{\mu\nu}=\varepsilon_\mu\varepsilon_\nu$ this becomes

$$\tfrac{1}{2}(p_\mu\varepsilon_\nu-p_\nu\varepsilon_\mu)(p_\alpha\varepsilon_\beta-p_\beta\varepsilon_\alpha)\,.$$

In the mathematical literature one introduces "curvature-two forms"; this amounts to absorbing the $(p_\alpha\varepsilon_\beta-p_\beta\varepsilon_\alpha)$ in our kinematical factor R. The standard mathematical formulae are then written in terms of "eigenvalues" of the "curvature-two forms"; this amounts to working with the eigenvalues x_i of $\frac{1}{2}\sum_j(p^{(j)}\varepsilon_\nu^{(j)}-p_\nu^{(j)}\varepsilon_\mu^{(j)})$, as we have done. The connection of our results with the index theorem in $4k+4$ dimensions will become clearer in sect. 11*.

In sects. 7 and 8 we will calculate the gravitational anomalies due to fields of spin $\frac{3}{2}$ and due to antisymmetric tensor fields. Happily, the tricks we have used in the spin-$\frac{1}{2}$ case will suffice, with a few minor modifications.

7. Gravitational anomalies for fields of spin $\frac{3}{2}$

Now we turn our attention to calculating the gravitational anomaly for fields of other spin. First we will consider the Rarita–Schwinger field $\psi_{\mu\alpha}$; μ is a vector index and α a spinor index. We wish to calculate the anomaly for a field that obeys a Weyl condition $\gamma_5\psi_\mu=-\psi_\mu$. We will not impose an additional Majorana condition; if this is done our answer must be divided by two.

In the quantization of the Rarita–Schwinger field, it is necessary to introduce several spin-$\frac{1}{2}$ Faddeev–Popov ghosts. Specifically, one needs two ghosts of the same chirality as ψ_μ and one of opposite chirality. Although this counting of ghosts sounds odd at first, it really has a simple explanation. Consider a physical propagating spin-$\frac{3}{2}$ particle of momentum k_μ. The constraints $k_\mu\psi^\mu=0$ and the gauge invariance under $\psi^\mu\to\psi^\mu+k^\mu\alpha$ (for any spin-$\frac{1}{2}$ field α) remove two spin-$\frac{1}{2}$ degrees of freedom of the same chirality as ψ_μ. The additional constraint $\gamma^\mu\psi_\mu=0$ removes one spin-$\frac{1}{2}$ degree of freedom of *opposite* chirality to ψ_μ. These conditions leave only a physical spin-$\frac{3}{2}$ particle.

As far as the anomalies are concerned, the effect of the ghosts is very simple. Two ghosts with the same chirality as ψ and one of opposite chirality contribute the same anomaly as one ghost of the same chirality as ψ. Hence we must simply compute the ψ_μ anomaly (using some non-singular lagrangian constructed by gauge fixing) and then subtract the spin-$\frac{1}{2}$ anomaly (already computed in sect. 6) for one ghost field with the same chirality as ψ_μ.

What non-singular lagrangian for ψ_μ may be used? The simplest lagrangian one might hope for would be a simple Dirac-like lagrangian:

$$\mathscr{L}=-\tfrac{1}{2}\bar{\psi}_\mu i\overleftrightarrow{\not{D}}\left(\frac{1-\gamma_5}{2}\right)\psi^\mu\,. \qquad (33)$$

* For gauge theories, such a connection was made by Goldstone (unpublished).

(The minus sign reflects a $(+------\cdots-)$ signature.) Actually, standard gauge fixing in the Rarita–Schwinger lagrangian leads not quite to (33) but to a lagrangian with some extra terms. But now we may make great use of the remarks at the end of sect. 2. The difference between (33) and the standard gauge-fixed Rarita–Schwinger lagrangian does not influence anomalies.

Now we can carry out an analysis similar to the one in sect. 6. Eq. (33) is almost equivalent to a theory of $4k+2$ decoupled spin-$\frac{1}{2}$ fields. The only difference arises because there is an extra term in the covariant derivative of ψ^μ: $D_\nu\psi^\mu = \partial_\nu\psi^\mu + \frac{1}{2}\omega_{\nu ab}\sigma^{ab}\psi^\mu + \Gamma^\mu_{\nu\alpha}\psi^\alpha$. The last term, proportional to the Christoffel symbol $\Gamma^\mu_{\nu\alpha}$, gives rise to additional interaction vertices.

Counting of gamma matrices similar to that which we carried out in sect. 6 shows that, to extract the anomaly in a diagram with $2k+2$ external gravitons, it is adequate to use for $\Gamma^\mu_{\nu\alpha}$ the linearized expression $\Gamma^\mu_{\nu\alpha} = \frac{1}{2}\eta^{\mu\sigma}(\partial_\nu h_{\sigma\alpha} + \partial_\alpha h_{\sigma\nu} - \partial_\sigma h_{\alpha\nu})$. The term proportional to $\partial_\nu h_{\sigma\alpha}$ can be discarded. It cancels upon including the covariant derivatives of ψ and $\bar\psi$ in $\bar\psi i\overleftrightarrow{\not{D}}\psi$ (essentially it cancels because for Majorana fermions, $\bar\psi_\sigma\gamma_\nu\psi_\alpha$ is antisymmetric in σ and α by Fermi statistics). Apart from the obvious generalization of eq. (23),

$$\mathscr{L}_1 = \tfrac{1}{4} i h^{\alpha\beta}\bar\psi_\mu\gamma_\alpha\overleftrightarrow{\partial}_\beta\left(\frac{1-\gamma_5}{2}\right)\psi^\mu\,,$$

$$\mathscr{L}_2 = \tfrac{1}{16} i(h_{\lambda\alpha}\partial_\sigma h_{\tau\alpha})\bar\psi_\mu\Gamma^{\sigma\lambda\tau}\left(\frac{1-\gamma_5}{2}\right)\psi^\mu\,, \tag{34}$$

the new interaction vertex is

$$\mathscr{L}_3 = \tfrac{1}{2} i(\partial_\sigma h_{\alpha\nu} - \partial_\alpha h_{\sigma\nu})\bar\psi^\sigma\gamma^\nu\psi^\alpha\,. \tag{35}$$

Now we carry out an analysis similar to the discussion in sect. 6. For diagrams with $2k+2$ external gravitons coupled to arbitrary vertices of $\mathscr{L}_1$, $\mathscr{L}_2$, or $\mathscr{L}_3$, the Dirac algebra leads to the same kinematical factor $R(\varepsilon_i, p_j)$ defined in sect. 6. After eliminating the Dirac algebra in this way, we are left with a theory of a boson ϕ^μ which in this case has spin one because it inherits the vector index carried by ψ^μ.

As before, in the effective boson theory, the vertices in $\mathscr{L}_1$ and $\mathscr{L}_2$ represent the minimal interaction of ϕ^μ with an electromagnetic field. What about $\mathscr{L}_3$? It has a very simple interpretation – it describes the magnetic moment of ϕ^μ. This arises as follows. For interaction with a graviton of momentum p_μ and polarization $\varepsilon_{\mu\nu} = \varepsilon_\mu\varepsilon_\nu$, (35) becomes $\frac{1}{2}(p_\mu\varepsilon_\nu - p_\nu\varepsilon_\mu)\bar\psi^\mu\not{\varepsilon}\psi^\nu$. The factor of $\not{\varepsilon}$ disappears in doing the Dirac algebra and passing from ψ^μ to an effective boson ϕ^μ. So $\mathscr{L}_3$ reduces to $\frac{1}{2}iF_{\sigma\nu}\phi^{*\sigma}\phi^\nu$ in the effective boson theory.

The effect of this is that the anomaly in the vector-spinor loop is

$$J = 2^{2k+1} i M^2 R(\varepsilon^{(i)}, p^{(j)})\tilde Z(\varepsilon^{(i)}, p^{(j)})\,, \tag{36}$$

where $\tilde Z$ is the effective action for a charged vector meson interacting with the constant electromagnetic field

$$F_{\mu\nu} = -i \sum_{j=0}^{2k+1} (p_\mu^{(j)} \varepsilon_\nu^{(j)} - p_\nu^{(j)} \varepsilon_\mu^{(j)}) .$$

The effective hamiltonian for the charged vector meson is (with now a euclidean signature) defined by

$$H\phi_\mu = -(\partial_\sigma + \tfrac{1}{4}iA_\sigma)^2 \phi_\mu + \tfrac{1}{2}iF_{\mu\nu}\phi_\nu .$$

As in sect. 6,

$$\tilde{Z} = \mathrm{Tr} \ln H = -\int_0^\infty \frac{\mathrm{d}s}{s} \mathrm{Tr}\, \mathrm{e}^{-sH} . \tag{37}$$

The trace in (37) is simple because $H = H_1 + H_2$ where $H_1 = -\delta_{\mu\nu}(\partial_\sigma + \frac{1}{4}iA_\sigma)^2$ and $H_2 = \frac{1}{2}iF_{\mu\nu}$. For a constant field, H_1 commutes with H_2. Since H_1 acts only on spatial variables and H_2 acts only on the spin, the trace in (37) factorizes as $(\mathrm{Tr}\, \mathrm{e}^{-sH_1})_{\text{space}} \cdot (\mathrm{Tr}\, \mathrm{e}^{-sH_2})_{\text{spin}}$. The spatial trace in that product is the trace discussed in sect. 6. The spin trace is the trace of a constant finite-dimensional matrix; in the notation of eq. (28), this trace is easily seen to be

$$\sum_{j=1}^{2k+1} 2 \cosh x_j .$$

Evaluating (37) by analogy with the treatment of (27), and remembering to subtract the contribution of a spin-$\frac{1}{2}$ ghost, the anomaly for the spin-$\frac{3}{2}$ field is

$$I_{3/2} = -i \frac{1}{(2\pi)^{2k+1}} R(\varepsilon^{(i)}, p^{(j)}) \times \left(\prod_{i=1}^{2k+1} \frac{\frac{1}{2}x_i}{\sinh(\frac{1}{2}x_i)} \right) \left(\sum_{j=0}^{2k+1} 2 \cosh x_j - 1 \right) . \tag{38}$$

Again, (38) is to be expanded in a power series in the x_i, only the term of order $2k+2$ in the x_i being relevant.

8. The antisymmetric tensor field

Unlike fermion integrals, which are formal constructions, euclidean space integrals for Bose fields are real, honest integrals. For fermions it is possible to integrate over a field ψ without integrating over its complex conjugate ψ^*. For bosons, instead, one always integrates over both the fields and their complex conjugates. In any theory that has a covariant formulation, the bosonic integration variables form a representation of the O(n) euclidean symmetry group, and it is always a real representation, for the reason just stated. Moreover, for bosons the euclidean action (or at least its real part) must be positive definite, and the kinetic energy is usually a sum of squares $\int dx \sum \theta_i(x)^2$ where the θ_i (linear in derivatives of Bose fields) may have various labels. Such an expression can always be regularized in the manner $\sum \theta_i^2 \to \sum_i ((1 - D_\mu D^\mu/\Lambda^2)\theta_i)^2$ and so leads to an effective action free of anomalies.

The only apparent exception to this reasoning would arise in the case of a boson theory which is Lorentz covariant or generally covariant but does not have a covariant lagrangian. There is no obvious, general way to regularize such a theory or prove the absence of anomalies.

There seems to be only one known case of a covariant Bose field without a covariant lagrangian [17]. This is the case of a theory in $4k+2$ dimensions with a field $A_{\mu_1 \cdots \mu_{2k}}$ that is an antisymmetric tensor with $2k$ indices. The curl of such a field $F_{\mu_1\mu_2\cdots\mu_{2k+1}} = (\partial_{\mu_1} A_{\mu_2 \cdots \mu_{2k+1}} + \text{cyclic permutations})$ is an antisymmetric tensor with $2k+1$ indices. $A_{\mu_1 \cdots \mu_{2k}}$ is a gauge field; under a gauge transformation it changes as the curl of an object with one less index; but its curl F is gauge invariant. In Minkowski space such $F_{\mu_1 \cdots \mu_{2k+1}}$ can be constrained to obey a self-duality condition $F_{\mu_1 \cdots \mu_{2k+1}} = (1/(2k+1)!)\varepsilon_{\mu_1\mu_2\cdots\mu_{4k+2}} F_{\mu_{2k+2}\cdots\mu_{4k+2}}$. Together with the Bianchi identity, the self-duality condition serves as a covariant equation of motion. We will consider A and F to be real; if they are complex, the anomaly is twice as big.

In Minkowski space of $4k$ dimensions, the self-duality condition would necessarily contain a factor of i (or $-i$), and CPT would relate the self-dual field to an anti-self-dual field. However, this is not true in $4k+2$ dimension; the self-dual field is self-conjugate under CPT. The self-dual field appears in certain supergravity theories in six and ten dimensions.

It seems that although there are covariant field theories containing the self-dual field, these theories have no covariant lagrangian. This suggests that there might be an anomaly in the coupling of the self-dual field to gravity. Indeed, we can immediately see that this is the case in two dimensions. In two dimensions $k=0$ and the antisymmetric tensor of $2k$ indices is just a scalar field ϕ. In terms of light-cone variables $x^\pm = \sqrt{\tfrac{1}{2}}(x^0 \pm x^1)$, the self-duality condition is just $\partial_-\phi = 0$. Thus, the self-dual field corresponds to "half" of a massless scalar. A massless scalar σ in two dimensions, which obeys $\partial_+\partial_-\sigma = 0$, can be written $\sigma = \sigma_+ + \sigma_-$, where $\partial_+\sigma_- = \partial_-\sigma_+ = 0$; the self-dual field is σ_+.

In two dimensions, by bosonization of fermions [21], a real scalar field is equivalent to a complex spinor field. The real self-dual field ϕ or σ_+ corresponds to the positive chirality complex spinor studied in sect. 5 and has exactly the same anomaly.

Now let us consider the situation in more than two dimensions. Precisely because a covariant lagrangian is not known (and the coupling to gravity has not been worked out even in the light-cone approach of [17]), we will have to be pragmatic and invent suitable Feynman rules.

Our discussion below simplifies slightly if we work in euclidean space. When we use gamma $\gamma_{\mu_1}, \gamma_{\mu_2}, \ldots, \gamma_{\mu_N}$ matrices, they will be real, symmetric $2^{N/2} \times 2^{N/2}$ matrices ($N = 4k+2$) that obey $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}$. We define $\gamma_5 = i\gamma_1\gamma_2 \cdots \gamma_N$; it is antisymmetric and its square is unity. We also define $\Gamma_{\mu_1\mu_2\cdots\mu_n} =$

$(1/n!)(\gamma_{\mu_1}\gamma_{\mu_2}\cdots\gamma_{\mu_n} \pm \text{permutations})$. Note that the transpose of $\Gamma_{\mu_1\cdots\mu_n}$ is $\Gamma_{\mu_n\cdots\mu_1}$ and that $\operatorname{Tr}\Gamma_{\mu_1\cdots\mu_n}\Gamma_{\nu_n\cdots\nu_1} = 2^{N/2}(g_{\mu_1\nu_1}\cdots g_{\mu_n\nu_n} \pm \text{permutations})$.

The coupling of the graviton to the antisymmetric tensor field $A_{\mu_1\cdots\mu_{2k}}$ is normally $-\frac{1}{2}h^{\mu\nu}T_{\mu\nu}$, where the energy momentum tensor is normally

$$T_{\mu\nu}(F) = \frac{1}{(2k)!}F_{\mu\alpha_1\cdots\alpha_{2k}}F_{\nu\alpha_1\cdots\alpha_{2k}} - \frac{1}{2(2k+1)!}g_{\mu\nu}(F_{\alpha_1\cdots\alpha_k})^2 . \tag{39}$$

Notice that the energy-momentum tensor does not involve $A_{\mu_1\cdots\mu_{2k}}$ explicitly but only the gauge invariant curl $F_{\mu_1\cdots\mu_{2k+1}}$. This means that to construct Feynman diagrams with external gravitons we do not need to construct a propagator for the gauge field A. It is enough to know the gauge invariant free propagator of F:

$$\langle F_{\mu_1\cdots\mu_{2k+1}}(q)F_{\nu_1\cdots\nu_{2k+1}}(-q)\rangle = -\frac{q_{\mu_1}q_{\nu_1}}{q^2}g_{\mu_2\nu_2}\cdots g_{\mu_{2k+1}\nu_{2k+1}}$$

$$\pm \text{permutations of } \mu_i\nu_j . \tag{40}$$

Now, to deal with the self-dual field we will assume that it is correct to use the propagator (40) without modification, while modifying the energy-momentum tensor. We take the interaction with gravity to be $-\frac{1}{2}h^{\mu\nu}T_{\mu\nu}(\frac{1}{2}(F-i\tilde{F}))$, where $\tilde{F}_{\mu_1\cdots\mu_{2k+1}} = 1/(2k+1)!\,\varepsilon_{\mu_1\cdots\mu_{2k+1}\nu_1\cdots\nu_{2k+1}}\cdot F^{\nu_1\cdots\nu_{2k+1}}$. Thus, we permit both self-dual and anti-self-dual fields in propagators, but only self-dual fields can be emitted or absorbed at vertices, since the energy-momentum tensor is constructed from the self-dual part of F only.

In practice, when we actually compute diagrams, we will use the ordinary energy-momentum tensor at every vertex except one, and $T_{\mu\nu}(\frac{1}{2}(F-i\tilde{F}))$ at one vertex only. Since duality is conserved at vertices, in that

$$T_{\mu\nu}(F) = T_{\mu\nu}(\tfrac{1}{2}(F-i\tilde{F})) + T_{\mu\nu}(\tfrac{1}{2}(F+i\tilde{F})) , \tag{41}$$

and since the propagator also conserves duality, it suffices to project out the self-dual part of F at one vertex only.

We now have a well-defined prescription for loop diagrams, and we could proceed, for instance, to compute hexagon diagrams in 10 dimensions (fig. 11). But the algebra would be quite cumbersome.

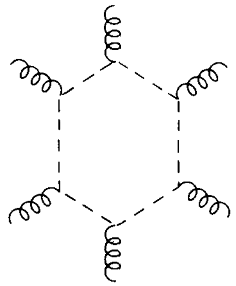

Fig. 11. A hexagon diagram in ten dimensions; the internal lines are antisymmetric tensor fields and the external lines are gravitons.

To achieve some simplification we introduce additional fields that are free of anomalies but permit a covenient reorganization of the algebra. We introduce not just the $2k$ rank tensor field $A_{\mu_1\cdots\mu_{2k}}$ but the whole complement of antisymmetric tensors A, A_μ, $A_{\mu_1\mu_2}$, $A_{\mu_1\mu_2\mu_3}, \ldots, A_{\mu_1\cdots\mu_N}$. (Recall $N=4k+2$ is the dimension of space–time.) And we define the gauge invariant curls

$$\begin{aligned} F &= 0\,, \\ F_\mu &= \partial_\mu A\,, \\ F_{\mu_1\mu_2} &= \partial_{\mu_1} A_{\mu_1\mu_3} + \text{cyclic permutations}\,, \\ &\vdots \\ F_{\mu_1\cdots\mu_N} &= \partial_{\mu_1} A_{\mu_2\cdots\mu_N} + \text{cyclic permutations}\,. \end{aligned} \tag{42}$$

The lagrangian for the antisymmetric fields are simple generalizations of the Maxwell lagrangian:

$$\mathscr{L} = \int \mathrm{d}^N x\,\sqrt{g} - \frac{1}{2\cdot n!} g^{\mu_1\nu_1} g^{\mu_2\nu_2}\cdots g^{\mu_n\nu_h} F_{\mu_1\cdots\mu_n} F_{\nu_1\cdots\nu_n}\,. \tag{43}$$

The energy-momentum tensor

$$T^{(n)}_{\mu\nu} = \frac{1}{(n-1)!} F_{\mu\alpha_1\cdots\alpha_{n-1}} F_{\nu\alpha_1\cdots\alpha_{n-1}} - \frac{1}{2\cdot n!} g_{\mu\nu}(F_{\alpha_1\cdots\alpha_n})^2\,, \tag{44}$$

and the free propagator

$$\langle F_{\mu_1\cdots\mu_n}(q) F_{\nu_1\cdots\nu_n}(-q)\rangle = -\frac{q_{\mu_1} q_{\nu_1} g_{\mu_2\nu_2}\cdots g_{\mu_n\nu_n}}{q^2} \pm \text{permutations of } \mu_i,\, \nu_j\,, \tag{45}$$

are simple generalizations of previous formulae.

The utility of introducing this host of new fields springs from a simple group theoretical fact. The tensor product of the spinor representation of O(N) with itself is precisely the direct sum of the nth rank tensor representation of O(N) for $0 \le n \le N$. Thus we can describe the whole collection (42) by a single field $\phi_{\alpha\beta}$ with two spinor indices α and β. We define

$$\phi_{\alpha\beta} = 2^{-N/4} \sum_{n=0}^{N} (\Gamma_{\mu_1\cdots\mu_n})_{\alpha\beta} F^{\mu_1\cdots\mu_n}\,. \tag{46}$$

The inverse formula is

$$F_{\mu_1\cdots\mu_n} = 2^{-N/4} (\Gamma_{\mu_n\cdots\mu_1})_{\beta\alpha} \phi_{\alpha\beta}\,, \tag{47}$$

as one may show by simple gamma matrix algebra. The point is that Feynman rules for $\phi_{\alpha\beta}$ are much more manageable than Feynman rules for the $F_{\mu_1\cdots\mu_n}$. Eqs. (45)

are reproduced by a simple equation

$$\langle\phi_{\alpha\beta}(q)\phi_{\gamma\delta}(-q)\rangle=\frac{1}{2q^2}((\gamma_5\not{q})_{\alpha\gamma}(\gamma_5\not{q})_{\beta\delta}+q^2\delta_{\alpha\gamma}\delta_{\beta\delta})\,, \tag{48}$$

for the $\phi_{\alpha\beta}$ propagator (recall $\gamma_5=i\gamma^1\gamma^2\cdots\gamma^N$). The total energy momentum tensor $T_{\mu\nu}=\sum_{n=0}^{N}T_{\mu\nu}^{(n)}$ (with $T_{\mu\nu}^{(n)}$ in (44)) is

$$T_{\mu\nu}=\tfrac{1}{4}\phi_{\alpha\beta}\phi_{\gamma\delta}((\gamma_\mu\gamma_5)_{\alpha\gamma}(\gamma_\nu\gamma_5)_{\beta\delta})+(\mu\leftrightarrow\nu)\,. \tag{49}$$

Eqs. (48) and (49) are much simpler than they may at first look. In (48) the second term lacks a pole at $q^2=0$ and can be discarded; it is a non-minimal term that does not affect the anomalies (though it is needed to reproduce (45)). The first term in (48) describes independent propagation of the two spinor indices α and β. The energy momentum tensor also propagates α and β independently. So the combinatories involves a product of two independent Dirac traces and is very simple.

We must generalize the duality operation $F\to\tilde{F}$ to the $\phi_{\alpha\beta}$ field. A suitable generalization, as one may see from (46), is

$$\phi_{\alpha\beta}\to(\gamma_5)_{\alpha\alpha'}\phi_{\alpha'\beta}\,. \tag{50}$$

For the $(2k+1)$-rank antisymmetric tensor, (50) is the old duality operation. For the other components, (50) is an operation that exchanges n-rank tensors with $(4k+2)$-n-rank tensors. The fact that (50) does not annihilate the n-rank tensors for $n\neq 2k+1$ is unwanted, in the sense that we do not wish to probe for anomalies in those fields, but it is harmless, in the sense that we know that the tensors with $n\neq 2k+1$ have no anomalies.

The Feynman rules (48) and (49) describe a theory with both self-dual and anti-self-dual fields propagating. We wish, instead, to compute the anomaly in a theory with only self-dual fields. As in our previous discussion we accomplish this as follows. At every vertex but one we use the energy-momentum tensor (49), but at one vertex, the anomalous vertex (indicated by a box in fig. 12) we use instead

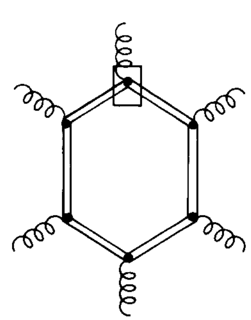

Fig. 12. The diagram of fig. 11, but with extra fields added and expressed in terms of a field $\phi_{\alpha\beta}$ with two spinor indices. $\phi_{\alpha\beta}$ is denoted as a double line to suggest independent propagation of the α and β indices. A box is drawn around the anomalous vertex.

a projected energy-momentum tensor

$$\tilde{T}_{\mu\nu} = \tfrac{1}{4}(\tfrac{1}{2}(1+\gamma_5)_{\alpha\alpha'}\phi_{\alpha'\beta}\cdot\tfrac{1}{2}(1+\gamma_5)_{\gamma\gamma'}\phi_{\gamma'\delta})((\gamma_\mu\gamma_5)_{\alpha\gamma}(\gamma_\nu\gamma_5)_{\beta\delta}) + (\mu\leftrightarrow\nu)\,. \quad (51)$$

We now evaluate diagram (12). At the ordinary vertices, the coupling is $-\frac{1}{2}h_{\mu\nu}T^{\mu\nu}$; the ith graviton has momentum $p_\mu^{(i)}$ and polarization $\varepsilon_{\mu\nu}^{(i)} = \varepsilon_\mu^{(i)}\varepsilon_\nu^{(i)}$. At the anomalous vertex, the coupling is $\varepsilon_\nu\partial_\mu\tilde{T}_{\mu\nu}$, where $\varepsilon_\nu = \varepsilon_\nu^{(0)}$ is the parameter of an infinitesimal coordinate transformation.

We regulate diagram (12) by the Pauli–Villars method. The regulator field $\tilde{\phi}_{\alpha\beta}$ has propagator

$$\langle\tilde{\phi}_{\alpha\beta}(q)\tilde{\phi}_{\gamma\delta}(-q)\rangle = \frac{1}{2(q^2+M^2)}((\gamma_5(\not{q}+iM))_{\alpha\gamma}(\gamma_5(\not{q}+iM))_{\beta\delta} + (q^2+M^2)\delta_{\alpha\gamma}\delta_{\beta\delta})\,. \quad (52)$$

Of course, in the regulated diagram, one may use naive manipulations and one need only keep the regulator loop. And the last term in (52) is irrelevant and may be dropped, for reasons mentioned earlier.

The actual evaluation of diagram (12) is rather simple*. Of the two Dirac traces, one contains at the anomalous vertex an insertion of $\not{p}$, p_μ being the graviton momentum. Let us call this trace the α trace and the other one the β trace. The α trace just gives $2^{2k+1}iM^2R(\varepsilon^{(i)}, p^{(j)})$ where R is the familiar kinematic factor of sect. 6. The remaining expression is very simple; it is one half the amplitude for a Dirac spinor of charge $\frac{1}{4}$ interacting with operators of momenta $p_\mu^{(i)}$ and polarizations $\varepsilon_\mu^{(i)}$. This should not be too surprising in view of our results of sects. 6 and 7. (Note that after dropping the irrelevant last term and performing the α trace, (52) reduces to $\frac{1}{2}\gamma_5$ times the standard Dirac propagator $(\not{q}+iM)/(q^2+M^2)$. Likewise after performing the α trace (49) becomes a standard Dirac vertex $\not{\varepsilon}$ times $\frac{1}{2}\gamma_5$.)

Our anomaly is hence

$$\tilde{I} = -\tfrac{1}{2}iM^2 2^{2k+1}R(\varepsilon^{(i)}, p^{(j)})Z\,, \quad (53)$$

where Z is the amplitude just mentioned for a charged spinor interacting with photons. As in our previous problems, Z can be computed as the amplitude for

* The interested reader should note the following points. One has, of course, a minus sign for the regulator loop. In matrix elements (or Feynman vertices) of the energy-momentum tensor one must include a factor of two from Bose statistics. The γ_5's in the propagators and vertices cancel harmlessly, leaving only γ_5's from projectors $\frac{1}{2}(1+\gamma_5)$. The γ_5 in the α trace goes into making the kinematic factor R, but the F in the β trace can be dropped (it gives terms that vanish as $M\to\infty$). So the $\frac{1}{2}(1+\gamma_5)$ in the β trace gives a factor of $\frac{1}{2}$. Another factor of $\frac{1}{2}$ occurs because the fields are real, but there is a factor of 2 from choosing which Dirac trace is the α trace and which is the β trace. Because of the two factors of $\frac{1}{2}$ and one factor of 2, we get eventually $\frac{1}{2}$ the amplitude for a charged Dirac particle in an external field.

propagation in the constant field $F_{\mu\nu}$ defined in sect, 7. So

$$\begin{aligned}Z &= \mathrm{Tr}\ln(i\not{D}+iM)\\ &= \tfrac{1}{2}\mathrm{Tr}\ln(-\not{D}^2+M^2)\\ &= \tfrac{1}{2}\mathrm{Tr}\ln(-D_\mu D^\mu+M^2+i\Gamma^{\mu\nu}F_{\mu\nu})\\ &= -\tfrac{1}{2}\int_0^\infty \mathrm{d}s\, \mathrm{e}^{-sM^2}\,\mathrm{Tr}\,\mathrm{e}^{-s(-D_\mu D^\mu+i\Gamma^{\mu\nu}F_{\mu\nu})}\,. \end{aligned} \tag{53}$$

(Here $\Gamma^{\mu\nu}=\frac{1}{2}[\Gamma^\mu,\Gamma^\nu]$.) For a constant field, $-D_\mu D^\mu$ and $i\Gamma^{\mu\nu}F_{\mu\nu}$ commute with each other, so the trace in (53) factorizes as a product of traces. The trace of $\mathrm{e}^{-s(-D_\mu D^\mu)}$ was evaluated in sect. 6, while in the notation of that section

$$\mathrm{Tr}\,\mathrm{e}^{-is\Gamma^{\mu\nu}F_{\mu\nu}}=2^{2k+1}\prod_{i=1}^{2k+1}(\cosh(\tfrac{1}{2}x_i))\,. \tag{54}$$

Evaluating the s integral in (53) as in sect. 6, we find that the anomaly $\tilde{I}$ of a real self-dual antisymmetric tensor field is

$$I=+\tfrac{1}{4}i\frac{1}{(2\pi)^{2k+1}}R(\varepsilon^{(i)},p^{(j)})2^{2k+1}\prod_{i=1}^{2k+1}\frac{\frac{1}{2}x_i}{\sinh(\frac{1}{2}x_i)}\cosh(\tfrac{1}{2}x_i)\,, \tag{55}$$

which, as always, is to be expanded to order $2k+2$ in the x_i. As far as the term of order $2k+2$ is concerned, (55) is equivalent to

$$I=\tfrac{1}{8}i\frac{1}{(2\pi)^{2k+1}}R(\varepsilon^{(i)},p^{(j)})\prod_{i=1}^{2k+1}\frac{x_i}{\tanh x_i}\,. \tag{56}$$

A special case of (56) can easily be tested. In two dimensions, by a bosonization argument mentioned earlier, the anomaly for a real self-dual scalar must equal that of a complex fermion of definite chirality. Hence as one can easily check, for $k=0$ the term of order x^2 in (56) must coincide with the term of order x^2 in eq. (32) for the spin-$\frac{1}{2}$ anomaly.

9. Mixed anomalies

Until now, our object has been to analyze anomalies that arise for theories with matter fields coupled to gravity only.

Higher-dimensional anomalies for theories with matter fields coupled to gauge fields only have been computed previously by several authors [7].

In this section, we will consider a more general problem that encompasses these two cases: we will calculate the anomalies for matter fields coupled to both gravity and gauge fields. Since (as far as is known) antisymmetric tensor fields cannot be consistently coupled to gauge fields, the relevant cases are the fields of spin $\frac{1}{2}$ or spin $\frac{3}{2}$.

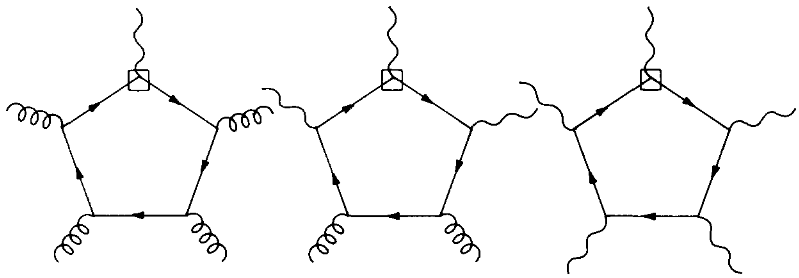

Fig. 13. The anomalous diagrams in eight dimensions. Gluons are wavy lines and gravitons are loopy lines. At one vertex (with a box around it) there is a projection operator $\frac{1}{2}(1-\gamma_5)$.

In any even number of dimensions $2n$, we will study diagrams with $n+1$ external boson lines, which may be gluons or gravitons. We will see that diagrams with r external gluons and $n+1-r$ external gravitons are always anomalous, as long as $n+1-r$ is even. For instance, in eight dimensions, the diagrams with five external gluons, three external gluons and two gravitons, or one external gluon and four gravitons are all potentially anomalous, depending on the gauge group (fig. 13). The cancellation of anomalies in higher-dimensional theories is much more difficult than in four dimensions; implications for some pseudo-realistic models will be described elsewhere [13].

So far, we have only discussed the particular case of diagrams with no external gluons and $n+1$ external gravitons. There is something very special about this case.

The triangle anomaly in four dimensions has two important but fundamentally different interpretations. If one external current is the generator of a global symmetry while the other two are coupled to gauge mesons, the anomaly represents the breakdown of the global symmetry in the presence of gauge fields. As such it can be related to the index theorem [22] for the four-dimensional Dirac operator. The anomalous part of the triangle can be deduced from the index theorem; calculating the triangle is a way to prove the index theorem.

The triangle anomaly has another interpretation, which has been stressed in this paper and is *not* directly related to any four-dimensional index theorem. If all three currents are coupled to gauge mesons, the anomaly represents a breakdown of gauge invariance.

The gravitational anomaly with external gravitons only has the second sort of interpretation. In gauge theories we can distinguish local symmetries from global symmetries; we can distinguish conserved currents coupled to gauge mesons from conserved currents that generate global symmetries. If precisely one of the currents is a global current, an anomalous diagram is related to an index theorem. In gravity we cannot distinguish a "global energy momentum tensor" from a "local energy-momentum tensor"; there is only one energy momentum then, and it couples to gravity. The anomalies evaluated in previous sections represent violation of general covariance. They cannot be interpreted as violations of a global conservation law in the presence of gravity. They cannot be derived from an index theorem or any known mathematical theorem*. This fact was one of the main motivations for the detailed calculations (and exposition) in this paper. By contrast, we now will examine anomalies in diagrams with gravitons *and* currents. By regarding one of the currents as the current of a global symmetry, these anomalies can be interpreted as breakdown of a global conservation law; and they can be deduced from (or used to prove) known index theorems. This was noted for the case of diagrams with external currents only in some of the papers of ref. [7].

We now turn to the detailed evaluation of the relevant diagrams, considering first the case of particles of spin $\frac{1}{2}$. Somewhere in the loop (fig. 13) there is a chirality projection operator $\frac{1}{2}(1-\gamma_5)$. We are really dealing, of course, with the one-loop diagram of a massive regulator field. At a vertex with gluon emission there is a factor

$$-i\gamma^\mu T^a_{\mathrm{L}}\,, \tag{57}$$

where T^a_{L} is the relevant group generator in the representation furnished by the left-handed fermions. Of course, the group theory factor associated with a given diagram is

$$\mathrm{Tr}\; T^{a_1}_{\mathrm{L}} T^{a_2}_{\mathrm{L}} \cdots T^{a_r}_{\mathrm{L}}\,. \tag{58}$$

After extracting this factor, what appears at a gluon vertex is just $-i\gamma^\mu$. At graviton vertices there appears instead

$$-\tfrac{1}{4}i\gamma_\mu(p+p')_\nu\,. \tag{59}$$

Now, recall that to deal with the graviton vertices our first step was to carry out the gamma matrix algebra. This depends only on the γ_μ factor in (59). Since that factor appears also in (57), the gamma matrix algebra goes through in the same way whether the external particles are gluons or gravitons. So all diagrams receive the same kinematic factor $R(\varepsilon^{(i)}, p^{(j)})$ discussed in sect. 5.

After removing this factor and the group theory factor (58), what remains of the graviton vertex is $-\frac{1}{4}i(p+p')^\mu$, corresponding to a particle of charge $\frac{1}{4}$ interacting with "photons"; what remains of a former gluon vertex is just $(-i)$, which one can think of as a vertex for absorption of a scalar σ. Of course, our propagators are now $i/(p^2-M^2)$, so we have an effective theory of charged scalars coupled to external photons and scalars (fig. 14).

We now wish to extract the limit as $M^2\to\infty$. Summing over permutations of gluons and gravitons, the one-loop diagrams give an effective action for $F_{\mu\nu}$ and σ. Terms involving derivatives of $F_{\mu\nu}$ or of σ vanish, on dimensional grounds, as

* M.F. Atiyah (private communication) pointed out that a relation between the gravitational anomaly in $4k+2$ dimensions and the index theorem in $4k+4$ dimensions should exist. This remark was based on certain properties of the diffeomorphism group in $4k\times 2$ dimensions. Our formulae for the anomalies are compatible with this idea.

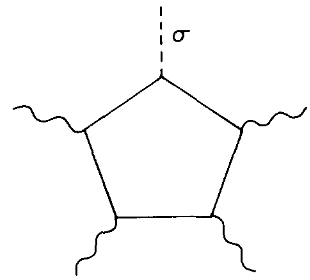

Fig. 14. After performing Dirac algebra, the Dirac propagators reduce to scalar propagators, the graviton vertices reduce to "photon" vertices, and the gluon vertices reduce to vertices for interaction with an effective scalar σ. The first diagram of fig. 13 is redrawn appropriately.

$M \to \infty$, so $F_{\mu\nu}$ can be treated as a constant electromagnetic field and the σ particles can be taken all to have zero momentum.

All dependence on momenta and polarizations of external gluons (which have been reduced to σ particles) is hence contained in $R(\varepsilon^{(i)}, p^{(j)})$. Since this factor is symmetric under permutations of the external lines, the group theory factor (58) must also be symmetrized, yielding what we will call $S\,\mathrm{Tr}\,(T_{\mathrm{L}}^{a_1} \cdots T_{\mathrm{L}}^{a_r})$, the symmetrized trace of the generators $T_{\mathrm{L}}^{a_1} \cdots T_{\mathrm{L}}^{a_r}$.

For emission of a zero-momentum scalar, the vertex $-i$ has a simple and well-known interpretation. It can be interpreted as resulting from differentiation with respect to M^2:

$$\frac{i}{p^2-M^2}(-i)\frac{i}{p^2-M^2} = \frac{\partial}{\partial M^2}\frac{i}{p^2-M^2}. \tag{60}$$

So a diagram with external photons and r external σ particles of zero momentum equals the rth derivative with respect to M^2 of a diagram with external photons only. But the amplitude for propagation in an external electromagnetic field only we have already evaluated. We may hence borrow our old results, just replacing (30) by

$$Z' = -S\,\mathrm{Tr}\,T_{\mathrm{L}}^{a_1} \cdots T_{\mathrm{L}}^{a_k}\left(\frac{\partial}{\partial M^2}\right)^r \int_0^\infty \frac{\mathrm{d}s}{s} \prod_{k=1}^{2k+1} \frac{\frac{1}{2}x_i}{4\pi \sinh(\frac{1}{2}x_i s)} \exp(-sM^2), \tag{61}$$

which now is to be expanded to order $\frac{1}{2}(n+2)-k$ in the x_i. The s integral can be done just as before, so

$$Z' = -S\,\mathrm{Tr}\,(T_{\mathrm{L}}^{a_1} \cdots T_{\mathrm{L}}^{a_r})\frac{1}{(4\pi)^{2k+1}}\frac{1}{M^2}\prod_{i=1}^{n/2}\frac{\frac{1}{2}x_i}{\sinh\frac{1}{2}x_i} \tag{62}$$

and the anomaly (since nothing else changes in the derivation of (32)) is

$$I_{1/2} = -S\,\mathrm{Tr}\,(T_{\mathrm{L}}^{a_1} \cdots T_{\mathrm{L}}^{a_r})\frac{i}{(2\pi)^{n/2}} R(\varepsilon^{(i)}, p^{(j)}) \prod_{i=1}^{n/2}\frac{\frac{1}{2}x_i}{\sinh\frac{1}{2}x_i}. \tag{63}$$

Eq. (63) is to be expanded to order $\frac{1}{2}(n+2)-r$ in the x_i, and expressed in terms of $\varepsilon^{(i)}$ and $p^{(j)}$ by rules explained in sect. 6.

Since (63) is even in each of the x_i, we see there is an anomaly only if $\frac{1}{2}(n+2)-r$ is even. There is potential trouble if the number r of external gluons is $r=\frac{1}{2}n+1$, $\frac{1}{2}n-1, \frac{1}{2}n-3, \cdots$. The condition for cancellation of anomalies (assuming only spin-$\frac{1}{2}$ fields are considered) is

$$S\,\mathrm{Tr}\; T_{\mathrm{L}}^{a_1} T_{\mathrm{L}}^{a_2} \cdots T_{\mathrm{L}}^{a_r} = 0\,, \tag{64}$$

for all such r and all $a_1 \cdots a_r$. For large n, (64) is very restrictive, since many values of r must be considered. For $r=\frac{1}{2}n+1$, this result was obtained in ref. [7].

In general, we must also include the contribution of possible charged fields of spin $\frac{3}{2}$. No new features arise in generalizing the conditions of sect. 7, and we get

$$I_{3/2} = -S\,\mathrm{Tr}\,(\tilde{T}_{\mathrm{L}}^{a_1} \tilde{T}_{\mathrm{L}}^{a_2} \cdots \tilde{T}_{\mathrm{L}}^{a_r}) \frac{i}{(2\pi)^{n/2}} R(\varepsilon^{(i)}, p^{(j)}) \times \prod_{i=1}^{n/2} \frac{\frac{1}{2}x_i}{\sinh \frac{1}{2}x_i} \times \left(\sum_{j=1}^{n/2} 2\cosh x_j - 1 \right), \tag{65}$$

where now $\tilde{T}_{\mathrm{L}}^{a}$ are the group generators for left-handed fields of spin $\frac{3}{2}$.

10. Global gravitational anomalies

So far we have considered anomalies that show up in perturbation theory. Such anomalies represent the lack of invariance of the effective action under infinitesimal general coordinate (or gauge) transformations.

Even if perturbative anomalies are absent, we must ask whether the effective action is invariant under coordinate or gauge transformations that cannot be reached continuously from the identity. Actually, in n dimensional euclidean space, we should restrict ourselves to coordinate transformations that approach the identity at infinity. Invariance under such coordinate transformations is needed for the internal consistency of a generally covariant theory, for reasons analogous to similar considerations in gauge theories. Let π be a hypothetical coordinate transformation that approaches the identity at infinity under which the effective action is not invariant. To be specific, suppose (since this is the case we will find) that the fermion integral changes sign under π in some theory. In this case, the theory is inconsistent because the euclidean path integral vanishes. The contribution to the path integral from any metric $g_{\mu\nu}$ would be exactly cancelled by the contribution of the conjugate metric $g_{\mu\nu}{}^{\pi}$ induced from $g_{\mu\nu}$ by the coordinate transformation π. Because π approaches the identity at infinity, one could not exclude $g_{\mu\nu}{}^{\pi}$ by a boundary condition. Moreover, $g_{\mu\nu}{}^{\pi}$ can be reached continuously from $g_{\mu\nu}$ (but not by means of coordinate transformations) by the interpolation $tg_{\mu\nu}+(1-t)g_{\mu\nu}{}^{\pi}$, $0 \leq t \leq 1$, so it is not possible to eliminate $g_{\mu\nu}{}^{\pi}$ by integrating over only "half" of field space.

Let us refer to coordinate transformations that are trivial at infinity and cannot be reached continuously from the identity as disconnected coordinate transformations. Finding such transformations is a difficult problem. It is known that in two or four dimensions there are no disconnected coordinate transformations, but this is unknown in three dimensions. The first example of a disconnected coordinate transformation was given by Milnor, in the six-dimensional case [23] (in constructing so-called exotic seven spheres). Typically, in n dimensions, for large n, there are many disconnected coordinate transformations. For instance, in six dimensions the group of coordinate transformations that are trivial at infinity has 28 components (the identity and 27 disconnected transformations); in eight dimensions this group has 8 components; in ten dimensions it has 992 components; in fourteen dimensions there are 16 256 [24].

There are three situations in which one might envisage global anomalies associated with disconnected general coordinate transformations:

(i) Matter fields in $4k+2$ dimensions which cannot have bare masses because they transform in a complex representation of $O(4k+2)$.

(ii) Theories in $8k$ or $8k+1$ dimensions with a single Majorana Fermi field (or an odd number of them). As discussed at the end of sect. 4, because of Fermi statistics a single Majorana field cannot acquire a mass in $8k$ or $8k+1$ dimensions; only in $8k$ or $8k+1$ dimensions does this phenomena occur.

(iii) Theories in which bare masses are possible.

We will not consider here cases of type (i). In these cases, infinitesimal anomalies appear in perturbation theory, as we have already discussed. The constraints from cancellation of infinitesimal anomalies are much more severe than any additional constraints that would come from global anomalies; however, we do not know if such additional constraints exist.

Regarding theories of type (iii), when bare masses are possible, Pauli–Villars regularization is possible. Since this regularization preserves general covariance, general covariance does not suffer from any local or global anomaly. However, in certain cases Pauli–Villars regularization violates parity, and in those cases there can be a global anomaly leading to breakdown of parity conservation. Although it is outside our main theme we will digress to consider this point.

In an even number of dimensions, Pauli–Villars regularization, if possible at all, always preserves parity. In an odd number of dimensions this is not the case. For instance, a Majorana fermion in $2+1$ dimensions may have a bare mass, but its bare mass violates parity. More generally, in $4k-1$ dimensions the sign of the mass term of a Majorana fermion is odd under parity. An *even* number of Majorana fermions can always receive parity-conserving bare masses (give positive bare masses to half of them and negative bare masses to the other half). With an *odd* number of Majorana fermions in $4k-1$ dimensions, parity conserving bare masses are not possible, and in this case a rather subtle global anomaly can ruin parity conservation.

If a bare mass is possible, the anomaly for any physical field ψ equals the anomaly of a very heavy regulator field χ that may be introduced. Since the whole effective action $\Gamma(\chi)$ of the very massive regulator χ is a local functional, any anomalous behavior of $\Gamma(\chi)$ (under coordinate transformations) can be cancelled by a local functional; therefore, any anomalous behavior of $\Gamma(\psi)$, the physically relevant effective action, can be cancelled by the same local counterterm. However, even if $\Gamma(\psi)$ conserves parity, it may be impossible to choose the local counterterm that cancels its anomalous variation to be parity conserving.

Since there is no trouble with *two* Majorana fermions, $e^{-2\Gamma(\psi)}$ (or $e^{-n\Gamma(\psi)}$ for even n) is generally covariant with no need for parity-violating counterterms. The worrisome possibility is that $e^{-\Gamma(\psi)}$ may change sign under a disconnected coordinate (or gauge) transformation π. If so, the anomalous behavior can be cancelled by adding to $\Gamma(\psi)$ the same local functional that cancels the anomalous behavior of $\Gamma(\chi)$.

$\Gamma(\chi)$ is local, and, like $\Gamma(\psi)$, it is infinitesimally generally covariant. The local functionals of the metric that are infinitesimally but not globally generally covariant are the Chern–Simons secondary characteristic classes Q_α. They are odd under parity and exist in $4k-1$ dimensions (in gauge theories the Chern–Simons classes exist in any odd number of dimensions). Our worry is that the Q_α may appear in $\Gamma(\chi)$. Because the Q_α are multivalued, their coefficients must ordinarily be integers in quantum field theory [25]. However, matter fields can modify this quantization. If $\Gamma(\chi)=\sum c_\alpha Q_\alpha$, with non-integral c_α, then to cancel the anomalous behavior of $\Gamma(\chi)$ – or more pertinently to cancel the equivalent anomaly of $\Gamma(\psi)$ – the coefficients of Q_α in the lagrangian must be $n_\alpha - c_\alpha$, with some integer n_α. In particular, the coefficient of Q_α in the lagrangian cannot vanish, and the quantum theory cannot conserve parity. There are non-trivial examples of this bizarre phenomenon, which is related to discussions of the η invariant in mathematics [26]. For instance, consider an SU(2) gauge theory in $2+1$ dimensions*. Ordinarily the coefficient of the Chern–Simons term (or "topological mass term") must be an integer [25]. If, however, a single Majorana SU(2) doublet is included, one finds (by reasoning similar to that in ref. [16]), that $e^{-\Gamma}$ is odd under a certain disconnected gauge transformation. To compensate for this the Chern–Simons coefficient must be a *half-integer*. It cannot vanish, and the quantum theory cannot conserve parity, contrary to what one would think classically. We do not know under what conditions such phenomena occur in general relativity.

Returning to our main theme, we now consider case (ii) in our previous catalogue of possibilities – the theory of a single Majorana Fermi field in $8k$ or $8k+1$ dimensions. In this case we will actually meet a breakdown of general covariance. We will consider only the case of fermions of spin $\frac{1}{2}$.

As discussed in sects. (2) and (4) (and in the remarks just concluded) for a Dirac field the fermion integral det $i\not{D}$ is perfectly invariant under general coordinate

* This point has been discovered independently and discussed in more detail by N. Redlich, ref. [25].

transformations. The difficulty arises in taking a square root to find the fermion integral $\sqrt{\det i\not{D}}$ of the Majorana field. The sign of the square root is potentially ambiguous.

Is there in $8k$ or $8k+1$ dimensions a disconnected coordinate transformation under which $\sqrt{\det i\not{D}}$ changes sign? Remarkably enough this question has already been answered by Hitchin [27] who proved the existence of such a transformation (this reference was pointed out by M. Atiyah).

Hitchin's interest was actually to prove the existence of riemannian metrics on an arbitrary manifold of $8k$ or $8k+1$ dimensions for which the Dirac operator has a zero eigenvalue. He did this as follows. In $8k$ dimensions (for example) the Dirac operator $\not{D}$ is a real, antisymmetric operator, just like the Dirac operator for SU(2) gauge fields in four dimensions. Its non-zero eigenvalues therefore occur in complex conjugate pairs. Hitchin proved the existence of a disconnected coordinate transformation π with the following property. In interpolating from $g_{\mu\nu}$ to $g_{\mu\nu}{}^{\pi}$ there are, for topological reasons, an odd number of "level crossings" in which an eigenvalue changes place with its complex conjugate (fig. 15). This guarantees the existence of zero eigenvalues somewhere between $g_{\mu\nu}$ and $g_{\mu\nu}{}^{\pi}$. As in ref. [15], it also means that $\sqrt{\det i\not{D}}$ is odd under the transformation π.

We conclude, therefore, that in $8k$ or $8k+1$ dimensions it is inconsistent to couple to gravity an odd number of Majorana fermions of spin $\frac{1}{2}$. We do not know if there is a similar problem for spin-$\frac{3}{2}$ fields, or whether, if so, spin-$\frac{1}{2}$ anomalies can be cancelled by spin-$\frac{3}{2}$ anomalies.

11. An alternative computation of the gravitational anomaly

In sects. 6–9 we have presented a diagrammatic calculation of the gravitational anomaly in $4k+2$ dimensions for Weyl fermions of spin $\frac{1}{2}$ and $\frac{3}{2}$ and for antisymmetric self-dual tensor gauge fields, as well as the combined gravitational and gauge

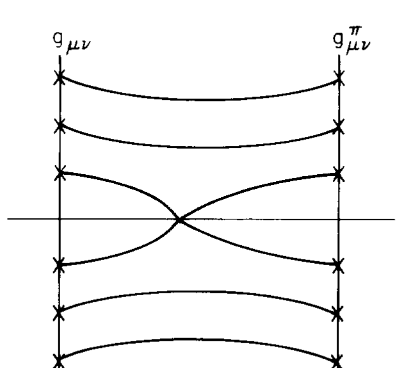

Fig. 15. An odd number of positive eigenvalues become negative in interpolating from $g_{\mu\nu}$ to $g^{\pi}_{\mu\nu}$. This is why $\sqrt{\det i\not{D}}$ is odd under π.

anomalies (figs. 9–14). In the detailed evaluation of the Feynman integrals, we arranged the trace over Dirac matrices and the Lorentz algebra in a convenient way so that the remaining integral could be reduced to the problem of evaluating the propagator of a particle in the presence of a constant external electromagnetic field. The fact that this method works for all the cases considered, suggests that there should be a method of computing the anomaly that exhibits the relation to the propagation of particles in constant external fields from the beginning.

In this section we will present an alternative way of evaluating the gravitational anomalies with or without gauge fields based on a generalization of a procedure first introduced by Fujikawa [28]. The basic idea of Fujikawa's procedure is to notice that the symmetries of the classical action are not necessarily symmetries of the measure in the functional integral which defines the graviton theory, and therefore classical symmetries may cease to be conserved at the quantum level. If one applies this method of computing anomalies to the usual axial anomaly [1], one gets a very clear connection between the anomaly and the Atiyah–Singer index theorem for the Dirac equation [22]. As will be fully explained below, this method proves to be very useful in relating (at least formally) the gravitational anomaly in $4k+2$ dimensions with the index theorem for certain operators in $4k+4$ dimensions, thus supporting the remark by M. Atiyah which was mentioned previously.

We illustrate the method we will use, by first computing the gravitational contribution to the axial anomaly for a fermion coupled to gravity in an arbitrary number of dimensions. Besides purely illustrative purposes, the results of this example will be very useful to us later.

Let $\psi(x)$ be a Dirac fermion defined on a $2n$-dimensional manifold M_{2n} with metric g_{ij}, and euclidean signature. The coupling of $\psi(x)$ to the gravitational field can be read off from (22)

$$\mathscr{L} = e\bar{\psi}(x) i\gamma^{\mu} D_{\mu}\psi . \tag{66}$$

(66) is invariant under a global axial U(1) transformation:

$$\psi(x) \to \mathrm{e}^{i\alpha\gamma_5}\psi(x) . \tag{67}$$

Under an infinitesimal space–time dependent chiral transformation (67) the action changes by

$$\delta S = \int \mathrm{d}x\, e\alpha(x) D_{\mu}(\bar{\psi}\gamma^{\mu}\gamma_5\psi) . \tag{68}$$

In order to check whether the axial current $\bar{\psi}\gamma_{\mu}\gamma_5\psi$ is still conserved at the quantum level, we consider the effective action

$$\mathrm{e}^{-\Gamma(g)} = \int \mathrm{d}\bar{\psi}\, \mathrm{d}\psi \exp\{-S(e, \bar{\psi}, \psi)\} , \tag{69}$$

and make a change of variables $\psi \to \psi' = \psi + i\alpha(x)\gamma_5\psi(x)$. As was pointed out in [28] the only term which could lead to anomalous contributions to the Ward identities is the measure in (69). The easiest way to define the measure in general, is to expand ψ and $\bar{\psi}$ in terms of the eigenfunctions of the Dirac equation

$$i\not{D}\psi_n = \lambda_n\psi_n ,$$

$$\psi = \sum_n a_n\psi_n , \qquad \bar{\psi} = \sum_n \psi_n^+(x)\bar{b}_n ,$$

so that the measure becomes $\prod_{n,m} d\bar{b}_m da_n$. Under the change of variables $\psi \to \psi + i\alpha(x)\gamma_5\psi$, the measure changes by a jacobian factor

$$\prod_{n,m} d\bar{b}_n da_m \to \left(\prod_{n,m} d\bar{b}_n da_m\right) \exp\left\{-2i \int \mathrm{d}x \sum_n \psi_n^+(x)\alpha(x)\gamma_5\psi_n(x)\right\} . \tag{70}$$

The minus sign on the exponent is due to Fermi statistics. If we let $\alpha(x)$ be a constant for simplicity, we have to evaluate

$$\sum_n \int (\mathrm{d}x)\psi_n^\dagger(x)\gamma_5\psi_n(x) . \tag{71}$$

This trace is clearly ill-defined. A simple way to define (71) is to use a gaussian cut-off [28]:

$$\begin{aligned} \sum_n \int \psi_n^\dagger(x)\gamma_5\psi_n(x) e \,\mathrm{d}x &\equiv \lim_{\beta\to 0} \int (\mathrm{d}x) e\psi_n^\dagger(x)\gamma_5\psi_n(x)\, \mathrm{e}^{-\beta\lambda_n^2} \\ &= \lim_{\beta\to 0} \mathrm{Tr}\, \gamma_5\, \mathrm{e}^{-\beta(i\not{D})^2} . \end{aligned} \tag{72}$$

The anomaly, if any, is given by the β-independent term of the right-hand side of (72). The procedure used in [28] to evaluate (72) becomes very cumbersome when trying to obtain the gravitational contribution to the axial anomaly in an arbitrary number of dimensions. The general philosophy of our procedure is to find a one-dimensional quantum mechanical system defined on the manifold M_{2n} such that its hamiltonian is $(i\not{D})^2$. Thus (72) becomes the partition function for an ensemble with the density matrix $\rho = \gamma_5 \exp[-\beta(i\not{D})^2]$ at temperature β^{-1}; or just $\mathrm{Tr}\,\rho$. Since $\mathrm{Tr}\,\rho$ has a functional integral representation, the evaluation of (72) is equivalent to the high-temperature expansion for the functional integral representation of $\mathrm{Tr}\,\rho$, which as will be shown below is a much simpler problem than the direct evaluation of (72).

The $(0+1)$-dimensional field theory we need is given by:

$$L = \tfrac{1}{2}g_{ij}(x)\frac{\mathrm{d}x^i}{\mathrm{d}\tau}\frac{\mathrm{d}x^j}{\mathrm{d}\tau} + \tfrac{1}{2}ig_{ij}(x)\psi^i\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\psi^j + \Gamma^i{}_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}\tau}\psi^l\right) , \tag{73}$$

where the x^i are the coordinates on M_{2n}, Γ^i_{jk} is the standard Cristoffel symbol constructed in terms of the metric $g_{ij}(x)$, and the $\psi^i(t)$ are one-component real fermionic variables. The form of (73) is suggested by the supersymmetric non-linear σ-model in two dimensions [29]. If we dimensionally reduce the $(1+1)$-dimensional σ-model to $0+1$ dimensions, we obtain

$$L=\tfrac{1}{2}g_{ij}(x)\frac{\mathrm{d}x^i}{\mathrm{d}\tau}\frac{\mathrm{d}x^j}{\mathrm{d}\tau}+\tfrac{1}{2}ig_{ij}(x)\psi^i_\alpha\left(\frac{\mathrm{d}}{\mathrm{d}t}\psi^j_\alpha+\Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}\tau}\psi^k_\alpha\right)+\tfrac{1}{4}R_{ijkl}\psi^i_1\psi^j_1\psi^k_2\psi^l_2\,,$$
$$\alpha=1,2\,, \tag{74}$$

(where R_{ijkl} is the curvature tensor on M_{2n}). If we impose the additional condition that $\psi^i_1=\psi^i_2=\sqrt{\tfrac{1}{2}}\psi^i$, we obtain (73). Before imposing this constraint, (74) is invariant under two supersymmetry transformations generated by two constant anticommuting real numbers ε_1, ε_2. After the constraint is implemented, (73) is still invariant under a single supersymmetry transformation with $\varepsilon_1=-\varepsilon_2=\varepsilon$. The supercharge is given after canonical quantization by $\sqrt{\tfrac{1}{2}}i\not{D}$, so that the hamiltonian of (73) becomes $H=\sqrt{\tfrac{1}{2}}(i\not{D})^2$. If we define a new set of fermion fields $\psi^a=e^a_i(x)\psi^i$, the canonical commutation relations which follow from (73) are:

$$\{\psi^a,\psi^b\}=\delta^{ab}\,, \tag{75}$$

and in terms of ψ^a, ψ^b, the canonical conjugate momentum to x^i is

$$p_i=g_{ij}(x)x^j+\tfrac{1}{4}i\omega_{iab}[\psi^a,\psi^b]\,. \tag{76}$$

Notice that (75) implies that the fermions generate a clifford algebra on M_{2n}, and that $\gamma_5=(-1)^F$; (i.e. $(-1)^F$ is the operator that anti-commutes with all Fermi fields), hence $\mathrm{Tr}\,\rho=\mathrm{Tr}\,(-1)^F\,\mathrm{e}^{-\beta H}$ which is the index for supersymmetric theories introduced in [30]. In fact, $\mathrm{Tr}\,\rho$ in this case is just the index of the Dirac equation [31]. The evaluation of $\mathrm{Tr}\,\rho$ is carried out in terms of its functional integral representation. Standard arguments imply [32]

$$\mathrm{Tr}\,\rho=\int_{\mathrm{PBC}}\mathrm{d}x(\tau)\,\mathrm{d}\psi(\tau)\exp\left\{-\int_0^\beta L_{\mathrm{E}}(\tau)\,\mathrm{d}\tau\right\}, \tag{77}$$

with bosons and fermions integrated over with periodic boundary conditions (PBC) with period β due to the presence of $(-1)^F$ in the trace. $L_{\mathrm{E}}(\tau)$ stands for the euclidean version of (73). In the $\beta\to0$ limit, (77) is dominated by constant paths $x^i=x^i_0$, $\psi^i=\psi^i_0$, which in this case are zero-action solutions of the classical equations of motion. Hence the leading $\beta\to0$ behavior of (77) is given by the quadratic term in the expansion of $L_{\mathrm{E}}(\tau)$ around the constant configurations (x^i_0,ψ^j_0). This expansion is greatly simplified if we use normal coordinates on M_{2n} [31, 33]. If $(\xi(\tau),\eta(\tau))$ are respectively the bosonic and fermionic non-constant small fluctuations around (x_0,ψ_0), the small-fluctuation lagrangian is

$$L^{(2)}_{\mathrm{E}}=\tfrac{1}{2}g_{ij}(x_0)\frac{\mathrm{d}\xi^i}{\mathrm{d}\tau}\frac{\mathrm{d}\xi^j}{\mathrm{d}\tau}-\tfrac{1}{4}R_{ijab}\psi^a_0\psi^b_0\xi^i\frac{\mathrm{d}\xi^j}{\mathrm{d}\tau}+\tfrac{1}{2}i\eta^a\frac{\mathrm{d}}{\mathrm{d}\tau}\eta^a\,.$$

Thus, the computation of the trace is reduced to the evaluation of a one-dimensional determinant. Since the fermionic fluctuations do not couple to the curvature to second order, they will cancel with the normalization factor (we normalize the trace with respect to the flat space case). Hence, the normalized trace is:

$$(\mathrm{Tr}\,\rho)_{\mathrm{norm}}=\frac{i^n}{(2\pi)^n}\int(\mathrm{d}x_0)(\mathrm{d}\psi_0)\,\frac{\det^{-1/2}(-\delta_{ab}\,\mathrm{d}^2/\mathrm{d}\tau^2+R_{ab}\,\mathrm{d}/\mathrm{d}\tau)}{\det^{-1/2}(-\delta_{ab}\,\mathrm{d}^2/\mathrm{d}\tau^2)}\,, \tag{79}$$

where $R_{ab}=\frac{1}{2}R_{abcd}\psi_0^c\psi_0^d$.

Despite the dependence of R_{ab} on fermion zero modes, we will treat the $R_{ab}\,\mathrm{d}/\mathrm{d}t$ term as part of the boson kinetic energy. In (79) we have redefined ξ^i to include the vierbein at x_0: $\xi^a=e_i^a(x_0)\xi^i$. Consequently the ξ kinetic term has the standard form, furthermore, the factor of $(2\pi)^{-n}$ comes from the standard Feynman measure for the bosonic degrees of freedom, and the factor of i^n is present because we are integrating over $2n$ real fermionic variables (the ψ_0's). Since R_{ab} is an antisymmetric $2n\times 2n$ matrix, we can skew diagonalize it, and call its eigenvalues x_α $\alpha=1,\dots,n$. Now (79) becomes

$$\begin{aligned}(\mathrm{Tr}\,\rho)&=\frac{i^n}{(2\pi)^n}\int_{\mathrm{M}_{2n}}\mathrm{d\,Vol}\int\prod_\alpha\prod_{n\geq 1}\left(1+\frac{(\frac{1}{2}ix_\alpha)^2}{\pi^2n^2}\right)^{-1}\mathrm{d}\psi_0\\&=\frac{i^n}{(2\pi)^n}\int_{\mathrm{M}_{2n}}\mathrm{d\,Vol}\int(\mathrm{d}\psi_0)\prod_\alpha\frac{(\frac{1}{2}ix_\alpha)}{\sinh(\frac{1}{2}ix_\alpha)}\,.\end{aligned} \tag{80}$$

Since the x_α are bilinear in the ψ_0's, and the polynomial appearing in the integrand of (80) is even under the interchange of $x_\alpha\to -x_\alpha$ for any number of α's, the number of ψ_0's in each monomial of the Taylor expansion of the integrand of (80) is a multiple of four, and thus the Grassmann integral will vanish unless the manifold has dimension $4k$. Formally, the constant anticommuting numbers ψ_0^a form a realization of the basis of 1-forms on the manifold, and the (ψ_0^a) integral just projects out the term proportional to $\psi_0^1\cdots\psi_0^{2n}$. More geometrically, let $R_{ab}=\frac{1}{2}R_{ab}e^a\wedge e^b$ be the curvature of the manifold referred to orthogonal frames, and let x_α, $\alpha=1,\dots,n$ be the set of skew eigenvalues of R_{ab}, then (80) can be rewritten as follows

$$(\mathrm{Tr}\,\rho)=\int_{\mathrm{M}_{2n}}\left(\prod_\alpha\frac{(x_\alpha/4\pi)}{\sinh(x_\alpha/4\pi)}\right)_{\mathrm{Vol}}\,. \tag{81}$$

The subscript "Vol" means that we have to pick out the term in the expansion of the integrand which is proportional to the volume form of M_{2n}. The function of the x_α appearing in (81) is the index density for the Dirac operator, and it is known in the mathematical literature as the Dirac genus, or $\hat{A}$ polynomial [6]. Its expansion in terms of the x_α is

$$A(\mathrm{M}_{2n})=1-\tfrac{1}{24}p_1(\mathrm{M})+\tfrac{1}{5760}(7p_1^2-4p_2)+\frac{1}{2^6 15120}(-16p_3+44p_1p_2-31p_1^3)+\cdots; \tag{82}$$

the $p_i(\mathrm{M})$ are known as Pontryagin classes and are defined by:

$$\det\left(1-\frac{1}{2\pi}R\right)=1+\frac{1}{(2\pi)^2}p_1+\frac{1}{(2\pi)^4}p_2+\frac{1}{(2\pi)^6}p_3+\cdots,$$

$$p_1(\mathrm{M})\equiv\sum_\alpha \omega_\alpha^2=-\tfrac{1}{2}\operatorname{Tr}R^2,$$

$$p_2(\mathrm{M})\equiv\sum_{\alpha<\beta}\omega_\alpha^2\omega_\beta^2=-\tfrac{1}{4}\operatorname{Tr}R^4+\tfrac{1}{8}(\operatorname{Tr}R^2)^2,$$

$$p_3(\mathrm{M})\equiv\sum_{\alpha<\beta<\gamma}\omega_\alpha^2\omega_\beta^2\omega_\gamma^2=-\tfrac{1}{6}\operatorname{Tr}R^6+\tfrac{1}{8}\operatorname{Tr}R^2\operatorname{Tr}R^4-\tfrac{1}{48}(\operatorname{Tr}R^2)^3. \tag{83}$$

Thus, the anomaly in four dimensions is given by $p_1(\mathrm{M})$ [5]:

$$\langle D_\mu J_5^\mu\rangle=-\frac{2i}{24(2\pi)^2}\tfrac{1}{8}R_{abcd}R^{ba}{}_{ef}\varepsilon^{cdef}=-\frac{2i}{384\pi^2}R\tilde{R}. \tag{84}$$

The factor of 2 in the numerator appears because we have been using a Dirac spinor. For a Weyl fermion, the result is half of (84).

The long exercise we have just gone through is not without sense. First it has permitted us to introduce our method and some useful notation, and second, the gravitational anomaly in $4k+2$ dimensions will be shown to be closely related to the axial anomaly in $4k+4$ dimensions.

In applying Fujikawa's method to the gravitational or mixed anomalies, care must be taken in interpreting the results. As it stands, this procedure does not generate the complete solution to the Wess–Zumino consistency conditions. The regulator we will use explicitly violates Bose symmetry because the gravitons have V−A couplings with the matter fields, not the vectorial couplings which appear in $i\not{D}$. However, if we only look for the leading term in the anomaly (the $n+1$ polygon in $2n$ dimensions), i.e. the term with the highest number of external momenta, the results that will be obtained are correct up to a factor of $1/(n+1)$ required to restore Bose symmetry in the leading term as explained thoroughly in sects. 2 and 6. The subleading terms could in principle be obtained using the Wess–Zumino consistency conditions. Conversely, we could start with the complete answer provided by Fujikawa's method which explicitly violates Bose symmetry and then use the Wess–Zumino consistency conditions to obtain the appropriate contact terms which need be subtracted in order to obtain the complete Bose symmetric solution of the consistency conditions. Thus in the final formulae for the anomaly which will follow, the curvature tensor should be understood to stand for $R_{\mu\nu\alpha\beta}=\partial^2_{\mu\alpha}h_{\nu\beta}+\partial^2_{\nu\beta}h_{\mu\alpha}-\partial^2_{\nu\alpha}h_{\mu\beta}-\partial^2_{\mu\beta}h_{\nu\alpha}$, and similarly, the gauge field strength $F^a_{\mu\nu}$ stands for $\partial_\mu A^a_\nu-\partial_\nu A^a_\mu$ as should be in the leading approximation. As will be shown below, this procedure gives the same answer as the diagrammatic calculation and thus provides an independent check on the computation of previous sections.

Let us now turn to the computation of the gravitational anomaly. Following the outline presented above, we first write down the transformation rule for spinors under an infinitesimal general coordinate transformation $x^i \to x^i + \eta^i(x)$. Up to a local Lorentz transformation:

$$\delta_\eta \psi = -\eta^i D_i \psi . \tag{85}$$

Since we are interested in the case when ψ is a Weyl fermion in $2n$ dimensions, the first problem we encounter is that, as explained in sect. 2, the operator $i\not{D}_{\rm L} = \frac{1}{2} i\not{D}(1-\gamma_5)$ is not self-adjoint. Thus in order to define the measure for the fermionic functional integral, we expand ψ in terms of the eigenfunctions of $(i\not{D}_{\rm L})^+(i\not{D}_{\rm L})$. We expand $\bar{\psi}$ in terms of the eigenfunctions of $(i\not{D}_{\rm L})(i\not{D}_{\rm L})^+$ [28]. With this definition of the functional integral and the measure, the jacobian induced by the infinitesimal coordinate transformation (85) is

$$g = \exp\left(\sum_n \int (\mathrm{d}x) e\psi_n^+(x)\eta^i(x) D_i\psi_n(x) - \sum_n \int (\mathrm{d}x) e\phi_n^+(x)\eta^i D_i \phi_n(x)\right). \tag{86}$$

Regularizing the trace as before:

$$\begin{aligned}\sum_n \int & (\mathrm{d}x) e\psi_n^+(x)\eta^i(x) D_i\psi_n(x) - \sum_n \int (\mathrm{d}x) e\phi_n^+ \eta^i D_i \phi_n(x) \\ &= \lim_{\beta\to 0} \int (\mathrm{d}x) e\left(\sum_n \psi_n^+(x)\eta^i D_i \,\mathrm{e}^{-\beta \not{D}_{\rm L}^+ \not{D}_{\rm L}} \psi_n - \sum_n \phi_n^+ \eta^i D_i \,\mathrm{e}^{-\beta \not{D}_{\rm L} \not{D}_{\rm L}^+} \phi_n\right) \\ &= \lim_{\beta\to 0} \mathrm{Tr}\; \eta^i D_i \gamma_5 \,\mathrm{e}^{-\beta(i\not{D})^2} ,\end{aligned} \tag{87}$$

which is very similar to (80) and can also be represented as a functional integral associated to (73). In terms of the fields defining the $(0+1)$-dimensional field theory (73), $\eta^i D_i$ can be represented as $i\eta_i(x)\,\mathrm{d}x^i/\mathrm{d}\tau$, and the trace (87) is just

$$\lim_{\beta\to 0} \int_{\rm PBC} \mathrm{d}x(\tau)\,\mathrm{d}\psi(\tau)\left(-\eta_i(x)\frac{\mathrm{d}x^i}{\mathrm{d}\tau}\right) \exp\left\{-\int_0^\beta L_{\rm E}(\tau)\,\mathrm{d}\tau\right\}. \tag{88}$$

As before, (88) is dominated in the $\beta\to 0$ limit by the constant configurations so that the only extra bit of information needed to evaluate (88) is the expansion of $\eta_i(x)\,\mathrm{d}x^i/\mathrm{d}\tau$ around a constant configuration (x_0, ψ_0). It is easy to see that to second order the expansion is given by $D_i\eta_j(x_0)\dot{\xi}^i\xi^j$. The computation is further simplified if we exponentiate $D_i\eta_j(x_0)\xi^i\dot{\xi}^j$, and at the end expand the result to first order in η^i. Once we exponentiate the only term which contributes is the antisymmetric part of $D_i\eta_j$, the symmetric part cancelling due to the periodicity of the boundary conditions. Thus, the only change with respect to the previous computation of the axial anomaly is that $R_{ab} = \frac{1}{2}R_{abcd}\psi_0^c\psi_0^d$ is substituted by $R_{ab} + D_a\eta_b - D_b\eta_a \equiv R'_{ab}$. If we denote the skew eigenvalues of R'_{ab} by x_α $\alpha = 1, \ldots, x_n$, and normalize the trace as before. The answer can be literally copied from (80):

$$\frac{i^n}{(2\pi)^n}\int_{M_{2n}} \mathrm{d}\,\mathrm{vol} \int \mathrm{d}\psi_0 \prod_\alpha \frac{\frac{1}{2}ix'_\alpha}{\sinh\left(\frac{1}{2}ix_\alpha^2\right)} . \tag{89}$$

Taylor expanding (89), the anomaly will be given by the term which is first order in η^i and contains $2n\psi_0$'s. It is not hard to convince oneself that such a term can only occur if $2n=4k+2$, which provides a nice check on the general arguments presented in sect. 4. To obtain the anomaly, we just expand the integrand in (89) to order $2k+2$ and then extract the term linear in η^i, which is exactly the same thing as writing the gravitational contribution to the axial anomaly in $4k+4$ dimensions with the substitution $R_{ab} \to R_{ab} + D_a\eta_b - D_b\eta_a$, and afterwards extracting the term proportional to η^i, as well as integrating over the constant fermionic variables. It is now a tedious exercise in elementary algebra to insert (82) in (89) and expand to the appropriate order. We will present the result in two ways. First we will give the combination of modified Pontryagin classes p'_i which contain the anomaly, and then we will explicitly display the anomaly in terms of the curvature tensor (by modified Pontryagin classes we mean the polynomials defined in (83) but with the substitution $R_{ab} \to R_{ab} + D_a\eta_b - D_b\eta_a$ understood). In 2 dimensions, the anomaly is contained in $\frac{1}{24} i p'_1$:

$$\int \mathrm{d}^2x\, e\eta^i D^j \langle T_{ij} \rangle = -\frac{i}{48\pi} \int \mathrm{d}^2x\, e D^i \eta^j R_{ijkl} \varepsilon^{kl} \,; \qquad ijkl = 1, 2\,. \tag{90}$$

For $d=6$, the corresponding coefficient is $-\frac{1}{5760} i(7p_1'^2 - 4p'_2)$:

$$\int \mathrm{d}^6x\, e\eta^i D^j \langle T_{ij} \rangle = -\frac{1}{2880(4\pi)^3} \int D^a \eta^b \varepsilon^{ijklmn} \times (5R_{baij} R^c_{\ jkl} R^d_{\ cmn} + 4R_b{}^c{}_{ij} R_c{}^d{}_{kl} R_{damn}) e\, \mathrm{d}^6x\,. \tag{91}$$

Finally, in ten dimensions the anomaly is contained in

$$+\frac{i}{2^6 15120}(-16p'_3 + 44p'_1p'_2 - 31p_1'^3)\,, \tag{92}$$

which yields

$$\begin{aligned} \int \mathrm{d}^{10}x\, e\eta^i D^j \langle T_{ij} \rangle = \frac{i}{(16\pi)^5 5670} \int \mathrm{d}^{10}x\, e D^a \eta^b & \\ \times (105 R_{abij} R_c{}^d{}_{kl} R_d{}^c{}_{mn} R_e{}^f{}_{pq} R_f{}^e{}_{rs} + 84 R_{abij} R_c{}^d{}_{kl} R_d{}^e{}_{mn} R_e{}^f{}_{pq} R_f{}^c{}_{rs} & \\ + 168 R_a{}^c{}_{ij} R_c{}^d{}_{kl} R_{dbmn} R_e{}^f{}_{pq} R_f{}^e{}_{rs} & \\ + 192 R_a{}^c{}_{ij} R_c{}^d{}_{kl} R_d{}^c{}_{mn} R_e{}^f{}_{pq} R_{fbrs}) \varepsilon^{ijklmnpqrs}\,. & \end{aligned} \tag{93}$$

It is clear that we only need to use the modified Pontryagin classes when looking for anomaly cancellations between fields of different spin. (We return to this subject in the next section.)

As pointed out in sect. 4, we have to calculate the anomaly for spin-$\frac{3}{2}$ Weyl fermions and self-dual antisymmetric tensors of rank $2k+1$ (in $4k+2$ dimensions).

It is quite clear from the above, that we need to extend the $(0+1)$-dimensional theory (73) in order to deal with the spin-$\frac{3}{2}$ case. To do this, let us consider in general a spinor field ψ_A on M_{2n} which contains some tensor indices A. If the index A is a vector index, we have the gravitino field; other type of tensor indices would imply that we are dealing with higher half-integer spin fields. Let $(T^{ab})_{AB}$ be the generators of the corresponding tensor representation of $SO(2n)$, $a, b = 1, 2, \ldots, 2n$; $A, B = 1, 2, \ldots, \dim T$. In the presence of the external gravitational field, the Dirac equation satisfied by ψ_A is the Dirac equation in the tensor product representation of the spinor representation and the tensor representation generated by T. Thus, if we want to calculate the gravitational anomaly, we have to extend (73) so that the hamiltonian of the new $(0+1)$-dimensional theory is the square of the Dirac operator in the representation of $SO(2n)$ defined by ψ_A. The only new ingredients required to achieve the desired generalization of (73) consist of including a new set of fermionic variables c^*_A, c_B which transform under $SO(2n)$ according to $(T^{ab})_{AB}$. The relevant lagrangian is [31]:

$$L = \tfrac{1}{2} g_{ij}(x) \frac{\mathrm{d}x^i}{\mathrm{d}\tau} \frac{\mathrm{d}x^j}{\mathrm{d}\tau} + \tfrac{1}{2} i \delta_{ab} \psi^a \left(\frac{\mathrm{d}}{\mathrm{d}t} \psi^b + \omega_{ic}^{\ \ b} \frac{\mathrm{d}x^i}{\mathrm{d}\tau} \psi^c \right)$$
$$+ i c^*_a \left(\frac{\mathrm{d}}{\mathrm{d}t} c_A + \tfrac{1}{2} i \omega_{iab} (T^{ab})_{AB} \frac{\mathrm{d}x^i}{\mathrm{d}\tau} c_B \right) + \tfrac{1}{4} i \psi^i \psi^j R_{ijab} c^*_A T^{ab}_{AB} c_B \,. \qquad (94)$$

If we were to calculate the axial anomaly, we would have to calculate $\mathrm{Tr}\, \gamma_5 \exp[-\beta(i\not{D})^2]$ in the $\beta \to 0$ limit with a constraint that will be mentioned shortly. The functional integral representation of the trace implies that $(x^i(\tau), \psi^j(\tau))$ are again integrated over with periodic boundary conditions with period β, while c_A, c^*_A are integrated over with antiperiodic boundary conditions.

There is a further constraint to be imposed on the trace, which is that the trace should run only over one-particle states of the c-fermions. This is because we are interested only in the anomaly corresponding to the T representation of $SO(2n)$ and not in any of its tensor products which are carried by the multiparticle states of c-fermions. Though this constraint may be difficult to impose in general, it is rather easy to implement in our case because we are only interested in the leading β behavior. We can briefly check that our procedure and conventions are correct by computing the spin-$\frac{3}{2}$ contribution to the axial anomaly in four dimensions which is known to be minus twenty-one times the spin-$\frac{1}{2}$ contribution [6]. In this case $(T^{ab})_{cd} = -i(\delta^a{}_c \delta^b{}_d - \delta^a{}_d \delta^b{}_c)$. Following the arguments presented for the spin-$\frac{1}{2}$ contribution at the beginning of this section, we have to compute the following trace

$$\mathrm{Tr}'\, \gamma_5\, \mathrm{e}^{-\beta(i\not{D})^2 s=3/2} = \int' \mathrm{d}x(\tau)\, \mathrm{d}\psi(\tau)\, \mathrm{d}c^*(\tau)\, \mathrm{d}c(\tau) \exp(-S_{3/2}) \,, \qquad (95)$$

where the apostrophe means that the computation is restricted to one-particle states of c and c^* and $s_{3/2}$ is the euclidean action associated to (94). In the high-temperature limit we again expand around constant configurations $(x_0, \psi_0, c = 0)$. Notice that there are no constant c configurations due to the boundary conditions, so that the lagrangian (94) is automatically second order with respect to the c's. Thus, in the small-β limit the terms involving the c's look like ordinary fermionic oscillators with a curvature dependent mass term, then the trace over one-particle states yields $\mathrm{Tr}\exp(\frac{1}{2}\psi_0^i\psi_0^j R^a_{ijb}(x_0))$, while the integral over $x(\tau)$ and $\psi(\tau)$ gives the same result as for the spin-$\frac{1}{2}$ axial anomaly. Including the ghost contributions discussed in sect. 7, the final result is:

$$\frac{i^n}{(2\pi)^n}\int \mathrm{d\,Vol}\int \mathrm{d}\,\psi_0(\mathrm{Tr}\,\mathrm{e}^R-1)\prod_\alpha \frac{\frac{1}{2}ix_\alpha}{\sinh(\frac{1}{2}ix_\alpha)}, \tag{96}$$

in (30), R represents the matrix $\frac{1}{2}R_{abcd}\psi_0^c\psi_0^d$. If we restrict (30) to a four-dimensional manifold, it follows after some algebra that the spin-$\frac{3}{2}$ contribution to the axial anomaly is twenty-one times the spin-$\frac{1}{2}$ contribution, and of opposite sign. It is now straightforward to compute the spin-$\frac{3}{2}$ contribution to the anomaly in the conservation of the energy momentum tensor. Up to a local Lorentz transformation, the change of ψ_A under an infinitesimal coordinate transformation $x^i\to x^i+\eta^i(x)$ is:

$$-\delta_\eta\psi_A=\eta^i D_i\psi_A+D_a\eta_b(T^{ab})_{AB}\psi_B\,. \tag{97}$$

Following the steps of the spin-$\frac{1}{2}$ case, the jacobian induced by (97) can be rewritten on a trace in term of the theory (94) with $(T^{ab})_{AB}$ being represented by $c^*{}_A(T^{ab})_{AB}c_B$. It then follows that the only change with respect to the computation of the axial anomaly is again the replacement of $\frac{1}{2}\psi_0^i\psi_0^j R_{ijab}$ by $\frac{1}{2}\psi_0^i\psi_0^j R_{ijab}+D_a\eta_b-D_b\eta_a$. Hence (96) with this substitution gives the desired result. The final answer

$$\frac{i^n}{(2\pi)^n}\int \mathrm{d\,Vol}\int \mathrm{d}\psi_0(\mathrm{Tr}\,\mathrm{e}^{R'}-1)\prod_\alpha \frac{\frac{1}{2}ix_\alpha}{\sinh(\frac{1}{2}ix_\alpha)}, \tag{98}$$

and R' is the matrix $R'_{ab}=R_{ab}+D_a\eta_b-D_b\eta_a$. Eq. (98) is again non-vanishing only when the space–time dimension $4k+2$. Writing out the anomaly in terms of the modified Pontryagin classes in two, six and ten dimensions, the result is:

$$A_{3/2}(d=2)=-\frac{23i}{48\pi}p_1'\,, \tag{99}$$

in two dimensions

$$A_{3/2}(d=6)=-\frac{i}{(2\pi)^3}\frac{1}{16}\left(\tfrac{55}{72}p_1'^2-\tfrac{49}{18}p_2'\right), \tag{100}$$

in six dimensions, and

$$A_{3/2}(d=10)=-\frac{i}{(2\pi)^5\cdot 2^6}\left(\tfrac{5}{336}p_1'^3-\tfrac{3}{28}p_1'p_2'+\tfrac{11}{21}p_3'\right), \tag{101}$$

if we wanted to obtain the final answer in terms of curvature tensors, we would have to substitute (82)–(83) in (99)–(101) and expand to first order in η^i. Since we will only be interested later on in anomaly cancellations, we will not display the final form of the anomaly in terms of curvature tensors.

With this method of computing the anomaly, we could in principle calculate the contribution to $\langle D^i T_{ij}\rangle$ due to higher half-integer spin fields $\frac{5}{2}, \frac{7}{2}, \ldots$. Since at the moment it is not clear how to quantize spin $\frac{5}{2}$ and higher in the presence of an external gravitational field [34], we won't discuss further this type of field.

Finally, we still have to consider the contribution to the anomaly coming from the self-dual antisymmetric tensor gauge field.

For this type of field, the methods presented so far in this section cannot be applied directly. The reason is that there is no generally covariant lagrangian leading to the equations of motion of the self-dual tensor field, as explained at length in sect. 8. Therefore there is no obvious $(0+1)$-dimensional field theory which could reproduce the anomaly for the self-dual tensor field. We will proceed in a manner similar to sect. 8. There, the anomalous polygon graph was calculated by noticing that the energy-momentum tensor only involves the gauge invariant field strength $F_{\mu_1\cdots\mu_{2k+1}}$, and that it naturally splits between two terms, one containing the self-dual part of F and the other containing the anti-self-dual part. Then the computation of the anomaly for the self-dual fields was accomplished by inserting in all but one vertex the unconstrained energy-momentum tensor, and the projected energy-momentum tensor in the remaining vertex.

After that, we included more tensor fields of lower and higher rank until we transformed the computation of the anomaly for a self-dual tensor field into an equivalent computation in terms of a bispinor field $\phi_{\alpha\beta}$ with positive chirality in each of its spinor indices. This procedure is legitimate because we know that the new fields added to $F_{\mu_1\cdots\mu_{2k+1}}$ in order to obtain $\phi_{\alpha\beta}$ do not contribute to the anomaly.

In order to apply Fujikawa's method, we consider the first-order formalism version of lagrangian (42). This simply means that we are integrating over both $A_{\mu_1\cdots\mu_{2k}}$ and $F_{\mu_1\cdots\mu_{2k+1}}$ as independent fields, and the lagrangian becomes:

$$\begin{aligned}\mathscr{L} = &\frac{2}{n!}\, g^{\mu_1\nu_1}\cdots g^{\mu_{2k+1}\nu_{2k+1}} F_{\mu_1\cdots\mu_{2k+1}} F_{\nu_1\cdots\nu_{2k+1}} \\ &-\frac{2}{n!}\, g^{\mu_1\nu_1}\cdots g^{\mu_{2k+1}\nu_{2k+1}} F_{\mu_1\cdots\mu_{2k+1}}\left(\frac{1}{n!}\,\partial_{\nu_1} A_{\nu_2\cdots\nu_{2k+1}} \pm \text{permutations}\right).\end{aligned}$$

It is easy to check that the equations of motion for $\mathscr{L}$ are the same as those following from (43). From this point of view, the integration measure is $[\mathrm{d}A\,\mathrm{d}F]$. Now we can split F into its self-dual and anti-self-dual parts, so that the integration measure becomes $[\mathrm{d}A\,\mathrm{d}F^+\,\mathrm{d}F^-]$. So far we are still working in Minkowski space, where the self-duality condition is real in $4k+2$ dimensions $(*^2=+1)$. If we perform an infinitesimal coordinate transformation, we will obtain three jacobians from the measure, one for each field A, F^+, F^-. Since $A_{\mu_1\cdots\mu_{2k}}$ is a tensor of rank $2k$, it furnishes a real representation of the Lorentz group and therefore it should not contribute to the anomaly. If we concentrate on the jacobian for the fields F^+, F^-, it is also clear that the anomalies that could arise from each of their jacobians cancel out because F^+ and F^- together generate a real representation of the Lorentz group. However, we argue in analogy with sect. 8, the anomaly for the self-dual will be fully contained in the jacobian generated by the F^+ field. If we perform an infinitesimal coordinate transformation $x^i \to x^i + \eta^i(x)$, the change in F^+ is given by

$$-\delta_\eta F^+_{a_1\cdots a_{2k+1}} = \eta^i D_i F^+_{a_1\cdots a_{2k+1}} + (D_a\eta_b)(T^{+ab})_{a_1\cdots a_{2k+1}, b_1\cdots b_{2k+1}} F^+_{b_1\cdots b_{2k+1}}, \tag{102}$$

where $(T^{+ab})_{a_1\cdots a_{2k+1}, b_1\cdots b_{2k+1}}$ are the matrices generating the self-dual $(2k+1)$-rank representation of the Lorentz group.

If we were to work in Minkowski space, it is easy to check that the jacobian factor generated by (102) is given by the following trace $\frac{1}{2}\mathrm{Tr}_{(2k+1)} * \delta\eta$, where the subscript $(2k+1)$ means that the trace is taken over $(2k+1)$-rank antisymmetric tensor. Since this trace is very ill-defined in Minkowski space, we can regularize it by going to euclidean space and introducing a gaussian cut-off as we did for the spin-$\frac{1}{2}$ and $\frac{3}{2}$ computations.

Since the duality condition becomes complex in euclidean space ($*^2 = -1$), we have to double the number of fields so that F becomes complex. We can easily take care of this by extending the trace to a complex trace, and dividing by 2 in order to account for the doubling of the number of degrees of freedom. Also, when rotating to euclidean space the duality operation $*$ becomes $i*$. Hence we have to calculate $\mathrm{Tr}_{(2k+1)}(i * \delta\eta)/4$. In order to regularize this trace, we notice that the unconstrained field $F_{\mu_1\cdots\mu_{2k+1}}$ satisfies the equations of motion $\Box F = 0$, where $\Box$ is the laplacian operator acting on antisymmetric tensors. Since we are working on euclidean space, the laplacian is a positive definite operator, and therefore we can compute the trace in the basis which make the laplacian diagonal. Thus we define the trace as the $\beta \to 0$ limit of $\mathrm{Tr}_{(2k+1)}(i * \delta\eta\, \mathrm{e}^{-\beta\Box})/4$. Before we proceed, it is worth pointing out that the laplacian commutes with the duality operation i.e. $\Box * = * \Box$, and that the equation $\Box F = 0$ means that F considered as a $2k+1$ form is harmonic, which implies among other things that $F_{\mu_1\cdots\mu_{2k+1}}$ can be rewritten as $\partial_{\mu_1} A_{\mu_2\cdots\mu_{2k+1}} \pm \text{perm}$. These two properties of $\Box$ make the choice of regularization very well suited for our computation. In order to simplify the trace we follow arguments of sect. 8. We add tensor fields of lower and higher rank so that the trace runs over tensors of all ranks: $0, 1, 2, \ldots, 2k+1, \ldots, 4k+2$. That this procedure is legitimate was explained in sect. 8 and will not be further discussed here. The final step in the evaluation of the trace, is to find a $(0+1)$-dimensional theory whose hamiltonian is the laplacian on forms of arbitrary rank. Happily, the answer to this question is already known [30]. It is given by the supersymmetric σ-model of eq. (74). In order to make the connection more clear, let us rewrite (74) in terms of complex fermions $\psi^i = \sqrt{\frac{1}{2}}(\psi^i + i\psi_2^i)$. After simple manipulations we get:

$$L = \tfrac{1}{2} g_{ij}(x)\dot{x}^i \dot{x}^j + \tfrac{1}{2} i g_{ij}(x)\psi^{*i}\frac{\overleftrightarrow{D}}{\mathrm{d}t}\psi^j + \tfrac{1}{4}R_{ijkl}\psi^{*i}\psi^{*j}\psi^k\psi^l, \tag{103}$$

after canonical quantization the fermions obey the usual anticommutation relations $\{\psi^i, \psi^{*j}\} = g^{ij}$, $\{\psi^i, \psi^j\} = 0$. From this point of view, the fermionic vacuum state is given by arbitrary functions over the manifold M_{4k+2} where the theory is defined. The states with one fermion are represented by the action of the fermionic creation operators ψ_i^* on the fermionic vacuum $\psi_i^*|\Omega\rangle$. Since ψ_i^* transforms like a vector under coordinate reparameterization of M_{4k+2}, one-fermion states transform like vectors. Similarly, states with two fermions are represented by a second rank antisymmetric tensor, and so on, until we reach the states with $4k+2$ fermions which correspond to totally antisymmetric tensors on M_{4k+2}. Furthermore, the duality operation which in spinor language is given by $\gamma_5 \otimes 1$ (see sect. 8), is here represented by a discrete symmetry. Q_5, which interchanges creation and annihilation operators [30]. Thus the trace we want to calculate is just $\mathrm{Tr}\, Q_5 \delta_\eta\, e^{-\beta H}$, where H is the hamiltonian associated to (74), (103). In terms of (74), using real fermions ψ_1^i, ψ_2^i, the Q_5 operation is $(\psi_1^i \to -\psi_2^i, \psi_2^i \to \psi_2^i)$, hence the functional integral representation for $\mathrm{Tr}\, Q_5 \delta_\eta\, e^{-\beta H}$ will be as in (77) but with (x^i, ψ_1^j) integrated over with periodic boundary conditions, whereas ψ_2^i has to be integrated over with antiperiodic boundary conditions. Finally, before we compute the trace, we have to identify δ_η in terms of the operators defining (74), (103). This is easily done in analogy with the spin-$\frac{3}{2}$ case. In eq. (96) we consider the infinitesimal coordinate transformation of a spinor field ψ with an extra index (A) valued in an arbitrary representation of the Lorentz group. If we take this extra index to be another spinor index, we get the form of the infinitesimal coordinate transformation of a bispinor, and the c-fermions are now replaced by the ψ_2^i. Thus the evaluation of $\mathrm{Tr}\, Q_5 \delta_\eta\, e^{-\beta H}$ is a rerun of the arguments which led to (98), where the trace in the integrand is carried out in the spinor representation. The final result is now

$$\begin{aligned} -\frac{1}{4}\frac{i^{2k+1}}{(2\pi)^{2k+1}} \int \mathrm{d}\psi_{10} \prod_\alpha \cosh\left(\tfrac{1}{2} i x'_\alpha\right) \left(\prod_\beta \frac{\frac{1}{2} i x'_\beta}{\sinh\left(\frac{1}{2} i x'_\beta\right)} \right) 2^{2k+1} \\ = -\frac{i^{2k+1}}{8(2\pi)^{2k+1}} \int \mathrm{d}\psi_{10} \prod_\alpha \frac{i x'_\alpha}{\tanh i x'_\alpha}\,, \end{aligned} \tag{105}$$

which agrees with eq. (56).

As in previous cases, we will write down the expansion of (103) in terms of the modified Pontryagin classes:

$$\begin{aligned} A(d=2) &= -\frac{i p'_1}{24(2\pi)}\,, \\ A(d=6) &= \frac{i}{5760(2\pi)^3}\left(16 p_1'^2 - 112 p'_2\right), \\ A(d=10) &= \frac{i}{967680(2\pi)^5}\left(-256 p_1'^3 + 1664 p'_1 p'_2 - 7936 p'_3\right). \end{aligned} \tag{106}$$

Finally, the combined gravitational and gauge anomalies can also be dealt with by the methods of this section in a very simple way. In fact the only non-trivial result necessary to carry out the computation is to generalize (94) so that the connection $\omega_i{}^a{}_b$ and the curvature R_{ijab} appearing in the last two terms of (94) can be substituted by the connection and curvature of an arbitrary gauge field defined on space–time.

For simplicity, we will only consider a Weyl fermion in $2n$ dimensions interacting with external gravitational and gauge fields. The extension of our results to include spin-$\frac{3}{2}$ Weyl fields is straightforward and will not be presented. Let G be an arbitrary gauge group, and assume that the fermion representation is generated by $(T^\alpha)_{AB}$ $\alpha=1,\dots,\dim \mathrm{G}$, $A, B=1,\dots,\dim T$. If we use Fujikawa's method [28], we have to find a $(0+1)$-dimensional field theory whose hamiltonian is $(i\not{D})^2$ where D_i is the covariant derivative with respect to the gravitational and gauge fields. In this case, by simple trial and error we can easily find the suitable modification of the lagrangian (94). The only change required is that now the c-fermions couple to the gauge connection $A_i^\alpha(x)$ and the gauge curvature F_{ij}^α and that the generators T^{ab} appearing in (94) are replaced by the gauge group generators $(T^\alpha)_{AB}$:

$$L=\tfrac{1}{2}g_{ij}(x)\frac{\mathrm{d}x^i}{\mathrm{d}\tau}\frac{\mathrm{d}x^j}{\mathrm{d}\tau}+\tfrac{1}{2}i\delta_{ab}\psi^a\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\psi^b+\omega_{ic}^{\ b}\frac{\mathrm{d}x^i}{\mathrm{d}\tau}\psi^c\right)$$
$$+ic_A^*\left(\frac{\mathrm{d}}{\mathrm{d}\tau}c_A+iA_i^\alpha(x)\frac{\mathrm{d}x^i}{\mathrm{d}\tau}(T^\alpha)_{AB}c_B\right)+\tfrac{1}{2}i\psi^i\psi^jF_{ij}^\alpha(x)c_A^*(T^\alpha)_{AB}c_B\,. \tag{107}$$

Imagine that $\omega_{ib}^a=0$, $g_{ij}=\delta_{ij}$. In order to check our conventions, we will compute the gauge contribution to the axial anomaly in $2n$ dimensions using our procedure first and then comparing the result with the standard Feynman graph calculation. The computation of the Adler–Bell–Jackiw anomaly [1] beyond 4-dimensions was carried out in [7], and we will borrow some notation from the last reference.

In the evaluation of the axial anomaly using our procedure we have to remember that when we write down the anomaly as a particular partition function for (107), we have to restrict the trace to one-particle states of the c-fermions for exactly the same reason as in the computation of the spin-$\frac{3}{2}$ contribution to the gravitational anomaly. If we rerun now the arguments which led to the computation of the spin-$\frac{3}{2}$ contribution to the axial anomaly with the obvious change that the index A is now an internal rather than a Lorentz index and that we are choosing for simplicity the space–time to be flat, we get for the axial anomaly:

$$+\frac{i^n}{(2\pi)^n}i\int(\mathrm{d}\psi_0)\,\mathrm{Tr}\,(\exp\tfrac{1}{2}i\psi_0^i\psi_0^jF_{ij}^\alpha T^\alpha)$$
$$=\frac{(-1)^n}{(2\pi)^n n!}\varepsilon^{i_1j_1\cdots i_nj_n}F_{i_1j_1}^{\alpha_1}(x_0)\cdots F_{i_nj_n}^{\alpha_n}(x_0)S\,\mathrm{Tr}\,T^{\alpha_1}\cdots T^{\alpha_n}\,, \tag{108}$$

where

$$S\,\mathrm{Tr}\,T^{\alpha_1}\cdots T^{\alpha_n}=\frac{1}{n!}\sum_{\substack{\mathrm{perm}\\ i_1\cdots i_n}}\mathrm{Tr}\,(T^{\alpha_{i1}}\cdots T^{\alpha_{in}})\,. \tag{109}$$

This result also follows from the index theorem for the Dirac equation [35], because as is well known, the axial anomaly is given by the local density of the Atiyah–Singer index theorem. If we also want to include the gravitational contribution to the anomaly, we just include the $\hat{A}(\mathrm{M})$ polynomial (82) in the integrand of (108) as follows from the arguments of the first part of this section. Thus the combined gauge and gravitational contribution to the axial anomaly in $2n$ dimensions is given by

$$\frac{i^n}{(2\pi)^n}\int(\mathrm{d}\psi_0)\,\mathrm{Tr}\exp\left(\tfrac{1}{2}i\psi_0^i\psi_0^jF_{ij}^\alpha T^\alpha\right)\prod_r\frac{\frac{1}{2}ix_r}{\sinh\left(\frac{1}{2}ix_r\right)}\,. \tag{110}$$

We are interested not only in the axial anomaly but also in the gauge invariance of the effective action. When we have both external and gravitational fields, we may expect anomalies in one-loop diagrams with both external gluons and gravitons. These are the anomalies we intend to calculate in the remainder of this section.

If we perform an infinitesimal gauge transformation

$$\delta\psi=i\eta_\alpha T^\alpha\psi\,,$$

and use the $(0+1)$-dimensional theory (107), the anomaly will be given by:

$$\lim_{\beta\to0}\mathrm{Tr}\,\gamma_5 i\eta_\alpha(c^*T^\alpha c)\,\mathrm{e}^{-\beta H}\,, \tag{111}$$

where as usual the trace is computed over one-particle states for the c-fermions. If we write (111) in terms of its functional integral representation, and exponentiate the term $\eta_\alpha c^*T^\alpha c$, we obtain

$$\lim_{\beta\to0}\mathrm{Tr}\,\gamma_5 i\eta_\alpha(c^*T^\alpha c)\,\mathrm{e}^{-\beta H}=\frac{i^n}{(2\pi)^n}\int(\mathrm{d}\psi_0)(\mathrm{Tr}\,\mathrm{e}^{i(F^\alpha+\eta^\alpha)T^\alpha})\hat{A}(M)\,,$$

$$F^\alpha=\tfrac{1}{2}F_{ij}^\alpha\psi_0^i\psi_0^j\,, \tag{112}$$

and the anomaly is extracted from (112) by expanding to first order in η^α. The expansion of (112) in $2n$ dimensions will clearly contain one term with n-gluons, then a term with $n-2$ gluons and 2 gravitons and so on. If instead we wanted to compute the variation of the effective action with respect to an infinitesimal general coordinate transformation, then we just have to set η^α to zero, and replace $\hat{A}(\mathrm{M})$ calculated with the standard curvature R_{abcd} by $\hat{A}'(\mathrm{M})$ calculated in terms of the modified curvature $R_{ab}+D_a\eta_b-D_b\eta_a$. If one wants to have an effective action that preserves gauge invariance, one would have to require the cancellation of the different symmetrized traces (109) appearing in the anomalous graphs between left- and right-handed representations. For left–right asymmetric theories, this constraint is highly non-trivial beyond four dimensions [7].

Explicit expressions for the combined anomalies can be obtained by combining (112) with the expression for $\hat{A}(\mathrm{M})$ in (82)–(84). In four dimensions we do not obtain any mixed anomaly unless the gauge group contains U(1) factors in which case there is a triangle anomaly with one external U(1) field and two external gravitons. This is the anomaly of refs. [5, 6] discussed at length in sect. 4.

We would like to remind the reader again that formulae (89), (98), (105), and (112) should be understood in terms of the leading term, i.e. $R_{\mu\nu\alpha\beta} = \partial_{\mu\alpha} h_{\nu\beta} + \partial^2_{\nu\alpha} h_{\mu\beta} - \partial^2_{\mu\beta} h_{\nu\alpha} - \partial^2_{\nu\alpha} h_{\mu\beta}$; $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A_\mu$. Non-leading terms should be obtained through application of the Wess–Zumino consistency conditions.

12. Cancellation of anomalies

We now turn, at last, to "phenomenological" considerations. We have seen that matter fields with chiral couplings to gravity exist only in $4k+2$ dimensions, and that such couplings give rise to anomalies. Of course, in standard phenomenology the individual quark and lepton multiplets have triangle anomalies in their couplings to gauge fields, but the anomalies cancel between multiplets in a non-trivial way. It is natural to ask whether a similar situation can occur with gravity. Although gravitational couplings of chiral fields of various spins give rise to anomalies, is it possible to cancel the anomalies between fields of different spin?

We have seen that in $4k+2$ dimensions, the gravitational anomalies are conveniently written as polynomials of order $2k+2$ in certain quantities x_i, $i = 1 \cdots 2k+1$. The spin-$\frac{1}{2}$, spin-$\frac{3}{2}$, and antisymmetric tensor anomalies are (apart from a factor common to the three cases)*

$$\hat{I}_{1/2} = \prod_{i=1}^{2k+1} \frac{\frac{1}{2}x_i}{\sinh \frac{1}{2}x_i},$$

$$\hat{I}_{3/2} = \left(\prod_{i=1}^{2k+1} \frac{\frac{1}{2}x_i}{\sinh \frac{1}{2}x_i}\right)\left(-1 + \sum_{j=1}^{2k+1} 2\cosh x_j\right),$$

$$\hat{I}_{\mathrm{A}} = -\tfrac{1}{8} \prod_{i=1}^{2k+1} \frac{x_i}{\tanh x_i}. \tag{114}$$

In $4k+2$ dimensions, one is to expand these formulae in powers of the x_i, keeping only the terms of order $2k+2$.

We observe that the formulas in (1) are invariant under permutations of the x_i and under $x_i \leftrightarrow -x_i$ for any i. So each term in their power series expansion has the

* By $I_{1/2}$ and $I_{3/2}$ we mean the anomalies of positive chirality complex Weyl fields. For a real Weyl field (possible in $8k+2$ dimensions) one must divide by two. I_{A} is the anomaly for a *self-dual* real antisymmetric tensor field. By a self-dual (rather than anti-self-dual) tensor we mean the representation that arises in combining two positive chirality spinors. For a complex self-dual tensor one must multiply I_{A} by two. The minus sign in I_{A} arises from the Bose (rather than Fermi) regulator loop; the factor of $\frac{1}{8}$ has a more complicated origin, which was explained in sects. 9 and 10.

same symmetries. To count the independent tensor structures appearing in the anomalies, it is enough to count the homogeneous polynomials of relevant degree and symmetry. In two dimensions ($k=0$), there is one x and one homogeneous second order polynomial x^2. In six dimensions there are three x_i and two linearly independent homogeneous fourth-order polynomials with the right symmetry, namely $\sum x_i^4$ and $(\sum x_i^3)^2$. In ten dimensions there are five x_i and three relevant polynomials, namely $\sum x_i^6$, $(\sum x_i^4)\cdot(\sum x_j^2)$, and $(\sum x_i^2)^3$. Cancellation of gravitational anomalies requires cancelling the coefficient of each dangerous operator. Beyond ten dimensions, the number of dangerous polynomials whose coefficients must cancel to avoid anomalies rapidly increases.

We could expand the anomalies as functions of $A_n=\sum_i x_i^{2n}$. Instead, following mathematical usage, we recall the origin of the x_i as eigenvalues of a matrix

$$R=\begin{pmatrix} 0 & x_1 & & & & & \\ -x_1 & 0 & & & & & \\ & & 0 & x_2 & & & \\ & & -x_2 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & x_{2k+1} \\ & & & & & -x_{2k+1} & 0 \end{pmatrix} \tag{115}$$

and we define polynomials p_i as follows. We note that $\det(1-R/2\pi)=\prod_i(1+x_i^2/(2\pi)^2)$. We write a power series expansion of the determinant:

$$\det(1-R/2\pi)=\sum_{n=0}^{\infty}\frac{p_n}{(2\pi)^{2n}}, \tag{116}$$

where p_n is a polynomial of order $2n$. Thus, $p_0=1$, $p_1=\sum x_i^2$, $p_2=\sum_{i<j}x_i^2x_j^2$, $p_3=\sum_{i<j<k}x_i^2x_j^2x_k^2$, etc. Every even symmetric polynomial of order $2n$ can be expanded in the p_m of $m\leq n$. We will express our results in this way*.

The first case that arises is two dimensions. Of course, this case is of mathematical interest only. In two dimensions

$$\hat{I}_{1/2}=-\tfrac{1}{24}p_1\,,\qquad \hat{I}_{3/2}=\tfrac{23}{24}p_1\,,\qquad \hat{I}_A=-\tfrac{1}{24}p_1\,. \tag{117}$$

Evidently, the anomaly can be cancelled in various ways. The fact that $\hat{I}_{1/2}=\hat{I}_A$ in two dimensions reflects the fact that a positive chirality fermion is equivalent to a right-moving scalar.

* We are grateful to P. Ginsparg for correcting a variety of numerical errors in an earlier version of this section.

For Kaluza–Klein theory our interest is in six or more dimensions, $k \geqslant 1$. In six dimensions power series expansion of eq. (114) yields

$$\hat{I}_{1/2} = \tfrac{1}{5760}(7p_1^2 - 4p_2)\,,$$
$$\hat{I}_{3/2} = \tfrac{1}{5760}(275p_1^2 - 980p_2)\,,$$
$$\hat{I}_A = \tfrac{1}{5760}(16p_1^2 - 112p_2)\,. \tag{118}$$

It may be seen that any two of these expressions are linearly independent, so that anomaly cancellation is possible only if all three spins are present. However, since there are three fields and only two independent anomalies (p_1^2 and p_2), there inevitably is a linear combination of these expressions that vanishes. The simplest non-trivial solution is $21\hat{I}_{1/2} - \hat{I}_{3/2} + 8\hat{I}_A = 0$. Thus a six-dimensional theory with 21 positive chirality spin-$\frac{1}{2}$ fields, one negative chirality gravitino, and eight self-dual antisymmetric tensor fields is free of anomalies. Although these numbers might seem clumsy, six-dimensional supergravity theories with this field content (modulo anomaly-free fields) do exist and might be of interest. It is a very favorable fact that the minimal solution only requires one gravitino; while there can be any number of spin-$\frac{1}{2}$ or antisymmetric tensor fields in six-dimensional supergravity, the number of gravitinos is necessarily $\leqslant 4$.

Turning now to ten dimensions, we find by power series expansion of (114)

$$\hat{I}_{1/2} = \tfrac{1}{967\,680}(-31p_1^3 + 44p_1p_2 - 16p_3)\,,$$
$$\hat{I}_{3/2} = \tfrac{1}{967\,680}(225p_1^3 - 1620p_1p_2 + 7920p_3)\,,$$
$$\hat{I}_A = \tfrac{1}{967\,680}(-256p_1^3 + 1664p_1p_2 - 7936p_3)\,. \tag{119}$$

Since there are three fields and three linearly independent anomalies, one would not *a priori* expect non-trivial calculation of gravitational anomalies to be possible in ten dimensions. But now we meet a real surprise, which is by far the most striking result of this paper. The expressions for $\hat{I}_{1/2}$, $\hat{I}_{3/2}$, and $\hat{I}_A$ in (119) are linearly dependent. In addition, the minimal solution is remarkably simple: $-\hat{I}_{1/2} + \hat{I}_{3/2} + \hat{I}_A = 0$. Thus, a ten-dimensional theory with one (complex) negative chirality spin-$\frac{1}{2}$ field, one (complex) positive chirality spin-$\frac{3}{2}$ field, and one (real) self-dual antisymmetric tensor is free of anomalies. What is more, modulo fields that do not contribute anomalies, this is precisely the field content of the chiral $n = 2$ supergravity theory in ten dimensions [11], which is the naive low-energy limit of one of the ten-dimensional supersymmetric string theories [12]. Since this theory cannot be coupled to supersymmetric matter multiplets, and cannot be extended to a theory with $n > 2$ supersymmetry (Nahm, ref. [11]), it appears to be the unique theory in ten dimensions with non-trivial cancellation of gravitational anomalies.

Are non-trivial anomaly cancellations possible beyond ten dimensions? In fourteen dimensions,

$$\hat{I}_{1/2}=\frac{1}{464\,486\,400}[381p_1^4-904p_1^2p_2+512p_1p_3+208p_2^2-192p_4]\,,$$

$$\hat{I}_{3/2}=\frac{1}{464\,486\,400}[6393p_1^4-42\,472p_1^2p_2-70\,144p_1p_3+102\,544p_2^2-94\,656p_4]\,,$$

$$\hat{I}_{A}=\frac{1}{464\,486\,400}[12\,288p_1^4-90\,112p_1^2p_2+290\,816p_1p_3+77\,824p_2^2 -1\,560\,576p_4]\,. \tag{120}$$

These expressions are linearly independent, so non-trivial anomaly cancellations are impossible in fourteen dimensions. (The situation becomes even worse if one considers that independent of anomalies consistent theories with massless spin-$\frac{3}{2}$ fields presumably do not exist in fourteen dimensions.)

A simple argument, similar to one in sect. 5, now shows that because non-trivial anomaly cancellation is impossible in fourteen dimensions, it is impossible in $14+4n$ dimensions for any $n\geq 1$. Consider a $(14+4n)$-dimensional manifold of topology $\mathrm{M}^{14}\times\mathrm{B}$, M^{14} being fourteen-dimensional Minkowski space and B a compact manifold of dimension $4n$ on which the Dirac, Rarita–Schwinger, and antisymmetric tensor equations have non-zero index. (For instance, B may be a product of n copies of K3.) An arbitrary chiral theory in $14+4n$ dimensions will reduce on $\mathrm{M}^{14}\times\mathrm{B}$ to a fourteen-dimensional chiral theory. Since this fourteen-dimensional chiral theory is necessarily anomalous, it follows that any chiral theory that might have been considered in $14+4n$ dimensions also has anomalies.

In conclusion, in any number of dimensions non-trivial cancellation of gravitational anomalies requires massless spin-$\frac{3}{2}$ fields and hence supergravity. In six dimensions there are various theories with non-trivial cancellation of gravitational anomalies. They require a fairly elaborate field content, but may be of interest. In ten dimensions the unique theory with such non-trivial cancellation is the chiral $n=2$ supergravity theory which is the low-energy limit of one of the superstring theories. Beyond ten dimensions non-trivial cancellations of gravitational anomalies does not occur. We will not explore here the phenomenological consequences of mixed gauge-gravitational anomalies.

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References

[1] S. Adler, Phys. Rev. 177 (1969) 2426; Lectures on elementary particles and quantum field theory, eds. S. Deser et al. (MIT, 1970);
J. Bell and R. Jackiw, Nuovo Cim. 60A (1969) 47;
R. Jackiw, in Lectures on current algebra and its applications (Princeton, 1972);
S.L. Adler and W. Bardeen, Phys. Rev. 182 (1969) 1517;
W.A. Bardeen, Phys. Rev. 184 (1969) 1848;
R.W. Brown, C.-C. Shi and B.-L. Young, Phys. Rev. 186 (1969) 1491;
J. Wess and B. Zumino, Phys. Lett. 37B (1971) 95;
A. Zee, Phys. Rev. Lett. 29 (1972) 1198

[2] J. Steinberger, Phys. Rev. 76 (1949) 1180;
J. Schwinger, Phys. Rev. 82 (1951) 664;
L. Rosenberg, Phys. Rev. 129 (1963) 2786;
R. Jackiw and K. Johnson, Phys. Rev. 182 (1969) 1459;
S. Adler and D.G. Boulware, Phys. Rev. 184 (1969) 1740;
S.L. Adler, B.W. Lee, S.B. Treiman and A. Zee, Phys. Rev. D4 (1971) 3497;
R. Aviv and A. Zee, Phys. Rev. D5 (1972) 2372;
M.V. Terentiev, JETP Lett. 14 (1971) 140;
A.M. Belavin, A.M. Polyakov, A.S. Schwarz and Yu.S. Tyupkin, Phys. Lett. 59B (1975) 85;
G. 't Hooft, Phys. Rev. Lett. 37 (1976) 8; Phys. Rev. D14 (1976) 3432;
C. Callan, R. Dashen and D.J. Gross, Phys. Lett. 63B (1976) 334;
R. Jackiw and C. Rebbi, Phys. Rev. Lett. 37 (1976) 172

[3] D.J. Gross and R. Jackiw, Phys. Rev. D6 (1972) 477;
C. Bouchiat, J. Iliopoulos and Ph. Meyer, Phys. Lett. 38B (1972) 519;
H. Georgi and S. Glashow, Phys. Rev. D6 (1972) 429

[4] G. 't Hooft, in Recent developments in gauge theories, G. 't Hooft et al., eds. (Plenum, New York 1980);
A.A. Ansel'm, JEPT Lett. 32 (1980) 138;
A. Zee, Phys. Lett. 95B (1980) 290;
Y. Frishman, A. Schwimmer, T. Banks and S. Yankielowicz, Nucl. Phys. B177 (1981) 157;
S. Coleman and B. Grossman, Nucl. Phys. B203 (1982) 205;
G.R. Farrar, Phys. Lett. 96B (1980) 273;
S. Weinberg, Phys. Lett. 102B (1981) 401;
C.H. Albright, Phys. Rev. D24 (1981) 1969;
I. Bars, Phys. Lett. 109B (1982) 73;
T. Banks, S. Yankielowicz and A. Schwimmer, Phys. Lett. 96B (1980) 67;
A. Schwimmer, Nucl. Phys. B198 (1982) 269

[5] R. Delbourgo and A. Salam, Phys. Lett. 40B (1972) 381;
T. Eguchi and P. Freund, Phys. Rev. Lett. 37 (1976) 1251

[6] S.W. Hawking and C.N. Pope, Nucl. Phys. B146 (1978) 381;
M.J. Perry, Nucl. Phys. B143 (1978) 114;
N.K. Nielsen, H. Romer and B. Schroer, Nucl. Phys. B136 (1978) 475;
N.K. Nielsen, M.T. Grisaru, R. Romer and P. van Nieuwenhuizen, Nucl. Phys. B140 (1978) 477;
A.J. Hanson and H. Romer, Phys. Lett. 80B (1978) 58;
R. Critchley, Phys. Lett. 78B (1978) 410;
S.M. Christensen and M.J. Duff, Phys. Lett. 76B (1978) 571; Nucl. Phys. B154 (1979) 301;
T. Eguchi, P.B. Gilkey and A.J. Hanson, Phys. Reports 66 (1980) 213

[7] P.H. Frampton and T.W. Kephart, Phys. Rev. Lett. 50 (1983) 1343, 1347;
P.K. Townsend and G. Sierra, Nucl. Phys. B222 (1983) 493;
B. Zumino, W. Yong-Shi and A. Zee (Univ. Washington preprint, 1983);
Y. Matsuki, and Y. Matsuki and A. Hill, OSU preprints, 1983

[8] J.H. Schwarz, Phys. Reports 89 (1982) 223, and references therein;
C.H. Ragadiakos and J.G. Taylor, Phys. Lett. 124B (1983) 201

[9] S.M. Christensen and S.A. Fulling, Phys. Rev. D15 (1977) 2088;
T.S. Bunch and P.C.W. Davies, Proc. Roy. Soc. A356 (1977) 569;

L.S. Brown, Phys. Rev. D15 (1977) 1469;
H.-S. Tsao, Phys. Lett. 69B (1977) 79;
J.S. Dowker and R. Critchley, Phys. Rev. D16 (1977) 3390
[10] E. Cremmer, B. Julia and J. Scherk, Phys. Lett. 76B (1978) 409;
E. Cremmer and B. Julia, Nucl. Phys. B159 (1979) 141;
A. Chamseddine, Nucl. Phys. B185 (1981) 403; Phys. Rev. D24 (1981) 3065
[11] W. Nahm, Nucl. Phys. B135 (1978) 149;
M.B. Green and J.H. Schwarz, Phys. Lett. 109B (1982) 444; 122B (1983) 143;
J. Schwarz and P. West, Phys. Lett. 126B (1983) 301;
P.S. Howe and P. West, King's College preprint (1983)
[12] M.B. Green and J.H. Schwarz, Nucl. Phys. B181 (1981) 502; B198 (1982) 252, 441; Phys. Lett. 109B (1982) 444;
J.H. Schwarz, Phys. Reports 89 (1982) 223
[13] E. Witten, Fermion quantum numbers in Kaluza–Klein theory, Princeton preprint, to appear
[14] B. de Witt, Phys. Reports 19 (1975) 295
[15] A. Salam and P.T. Matthews, Phys. Rev. 90 (1953) 690;
J. Schwinger, Phys. Rev. 93 (1954) 615
[16] E. Witten, Phys. Lett. 117B (1982) 324
[17] N. Marcus and J. Schwarz, Phys. Lett. 115B (1982) 111
[18] L. Brink, J.H. Schwarz and J. Scherk, Nucl. Phys. B121 (1977) 77;
N.S. Manton, Nucl. Phys. B193 (1981) 391;
G. Chapline and N. Manton, Nucl. Phys. B184 (1981) 391;
G. Chapline and R. Slansky, Nucl. Phys. B209 (1982) 461;
R. Coquereaux, Phys. Lett. 115B (1982) 389;
T. Kugo and P. Townsend, Nucl. Phys. B221 (1983) 357;
C. Wetterich, Phys. Lett. 110B (1982) 379; 113B (1982) 377; Nucl. Phys. B211 (1983) 117; B222 (1983) 20
[19] G. 't Hooft, Nucl. Phys. B75 (1974) 461
[20] J.S. Schwinger, Phys. Rev. 82 (1951) 664;
C. Itzykson and J.-B. Zuber, Quantum field theory (McGraw-Hill, 1980)
[21] S. Coleman, Phys. Rev. D11 (1975) 2088;
S. Mandelstam, Phys. Rev. D11 (1975) 3026
[22] M.F. Atiyah and I.M. Singer, Ann. of Math. 87 (1968) 485, 546; 93 (1971) 1, 119, 139;
M.F. Atiyah and G.B. Segal, Ann. of Math. 87 (1968) 531
[23] J. Milnor, Ann. of Math. (2) 64 (1956) 399
[24] S. Smale, Ann. of Math. (2) 74 (1961) 392
[25] S. Deser, R. Jackiw and S. Templeton, Phys. Rev. Lett. 48 (1982) 975; Ann. of Phys. 140 (1982) 372;
N. Redlich, MIT preprint, to appear
[26] M.F. Atiyah, V.K. Parodi and I.M. Singer, Proc. Camb. Philos. Soc. 77 (1975) 43; 78 (1975) 405; 79 (1976) 71
[27] N. Hitchin, Adv. Math. 14 (1974) 1
[28] K. Fujikawa, Phys. Rev. Lett. 42 (1979) 1195; 44 (1980) 1733; Phys. Rev. D21 (1980) 2848; D22 (1980) 1499 (E); D23 (1981) 2262;
M.B. Einhorn and D.R.T. Jones, U.M.-T.H. 83-3 preprint
[29] P. di Vecchia and S. Ferrara, Nucl. Phys. B130 (1977) 93;
E. Witten, Phys. Rev. D16 (1977) 2991;
D.Z. Freedman and P.K. Townsend, Nucl. Phys. B177 (1981) 282
[30] E. Witten, Nucl. Phys. B202 (1982) 253
[31] L. Alvarez-Gaumé, Harvard preprints HUTP-82/A029 and A035
[32] S. Cecotti and L. Girardello, Phys. Lett. 110B (1982) 39
[33] L. Alvarez-Gaumé, D.Z. Freedman and S. Mukhi, Ann. of Phys. 134 (1981) 85
[34] P.V. Nieuwenhuisen, Phys. Reports 68 (1981) 189
[35] R. Jackiw and C. Rebbi, Phys. Rev. D14 (1976) 517;
N.K. Nielsen, H. Römer and B. Schroer, Phys. Lett. 70B (1977) 445
[36] S. Fubini, R. Jackiw and A. Hanson, Phys. Rev. D7 (1973) 1732