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INSTABILITY OF THE KALUZA-KLEIN VACUUM*

Edward WITTEN

Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08544, USA

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It is argued that the ground state of the Kaluza-Klein unified theory is unstable against a process of semiclassical barrier penetration. This is related to the fact that the positive energy conjecture does not hold for the Kaluza-Klein theory; an explicit counter-example is given. The reasoning presented here assumes that in general relativity one should include manifolds of non-vacuum topology. It is argued that the existence of elementary fermions (not present in the original Kaluza-Klein theory) would stabilize the Kaluza-Klein vacuum.

In the Kaluza-Klein approach to unification of gauge fields with general relativity [1] – as seen from a modern point of view [2] – the starting point is general relativity in $4+n$ dimensions. But instead of assuming the ground state to be M^{4+n} (Minkowski space in $4+n$ dimensions) one assumes it to be a product $M^4 \times B$, where M^4 is ordinary four-dimensional Minkowski space and B is a suitable compact manifold. The low-energy physics is then obtained by expanding around the presumed ground state $M^4 \times B$. Apart from a massless graviton, one obtains among the small oscillations around $M^4 \times B$ massless gauge mesons associated with all of the symmetries of B.

With a suitable choice of B one might hope to get a realistic model of particle physics. A discussion of some of the possibilities has been given in ref. [3].

To pursue a program such as this, it is important to have some criteria for determining whether something of the form $M^4 \times B$ is really a reasonable candidate as the ground state of a given theory. We cannot answer this question in full because of our lack of understanding of dynamics – particulary since general relativity in four or more dimensions is unrenormalizable. However, it is possible to impose the requirement that $M^4 \times B$ should be stable at the classical and semiclassical level. This leads to non-trivial conditions; the purpose of this paper is to begin to explore those conditions.

In particular, we will examine the original five-dimensional theory of Kaluza and Klein. It will be argued that the ground state of the original Kaluza-Klein theory, although stable classically, is unstable against a semiclassical decay process.

In the original Kaluza-Klein theory, it is assumed that the ground state of five-dimensional relativity is not five-dimensional Minkowski space M^5, but rather

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is $M^4 \times S^1$, the product of four-dimensional Minkowski space with a circle S^1. Being flat, $M^4 \times S^1$ satisfies the classical Einstein equations regardless of what radius R is assumed for the circle. Because of the U(1) symmetry of the circle, this theory possesses a single abelian massless gauge meson.

Is it possible that $M^4 \times S^1$ could be the ground state of five-dimensional relativity? One cannot answer this question simply by comparing the energy of M^5 to the energy of $M^4 \times S^1$. Both spaces have zero energy. However, the definition of energy in general relativity depends on the boundary conditions, and because $M^4 \times S^1$ and M^5 have different asymptotic behavior, a comparison between the zero energy of $M^4 \times S^1$ and the zero energy of M^5 would be meaningless. There is a way to assess the stability of $M^4 \times S^1$ by an argument involving energy considerations, but it is slightly subtle; we will return to the point later.

Instead of trying to compare $M^4 \times S^1$ to Minkowski space, let us see what we can say about the properties of $M^4 \times S^1$ itself.

The first test of the stability of a space is to ask whether the space is stable, classically, against small oscillations. The Kaluza-Klein vacuum passes this test. The small oscillations around $M^4 \times S^1$ consist of several massless states (a graviton, a photon, and a Brans-Dicke scalar) and an infinite number of massive, charged modes of spins zero, one and two. There are no exponentially growing modes with imaginary frequencies.

Even if a state is stable against small oscillations, it may be unstable at the semiclassical level. This can occur if the state is separated by only a finite barrier from a more stable state. It will then be unstable against decay by semiclassical barrier penetration. The theory of such semiclassical instabilities in field theory has been developed in detail in the last few years [4]. Since it is not straightforward to assess the stability of $M^4 \times S^1$ by an argument based on energetics, let us simply look for evidence for a semiclassical instability.

To look for a semiclassical instability of a putative vacuum state, one looks for a "bounce" solution of the classical euclidean field equations. This is a solution which asymptotically, at infinity, approaches the putative vacuum state. If the solution is unstable (the determinant of small oscillations has negative modes), then the gaussian integral around this solution gives an imaginary part to the energy of the vacuum state, indicating an instability. Such considerations were first applied to gravity by Perry [5].

Thus, to assess the stability of Minkowski space itself at the semiclassical level, one would look for a solution of the classical Einstein equations which at infinity is asymptotic to flat, infinite euclidean space. Such a solution, if it possessed negative modes for small oscillations, would indicate the instability of Minkowski space. However, it has been proved [6, 7] that such solutions do not exist, at least in the absence of matter fields.

How would we search for a semiclassical instability of the Kaluza-Klein vacuum? We first analytically continue the Kaluza-Klein vacuum to euclidean space (i.e., to a positive signature) so that the metric is

$$ds^2 = dx^2 + dy^2 + dz^2 + dt^2 + d\phi^2 . \tag{1}$$

Here x, y, z and t run from $-\infty$ to $+\infty$, but ϕ is a periodic variable, which runs from 0 to $2\pi R$, R being the radius of the Kaluza-Klein circle. Introducing polar coordinates in the space spanned by x, y, z, and t, this can equivalently be written

$$ds^2 = dr^2 + r^2\, d\Theta^2 + d\phi^2 , \tag{2}$$

where r runs from 0 to ∞ and $d\Theta^2$ is the line element of the three sphere. We now want to look for a solution of the classical Einstein equations with the same asymptotic behavior as (2). Actually, such a solution is [8]

$$ds^2 = \frac{dr^2}{1-\alpha/r^2} + r^2\, d\Theta^2 + \left(1 - \frac{\alpha}{r^2}\right) d\phi^2 . \tag{3}$$

(One may recognize this as the five-dimensional Schwarzschild solution, analytically continued, but the interpretation here will be different.) For any value of α, this satisfies the classical Einstein equations except at the dangerous point $r = \sqrt{\alpha}$. Now we must discuss the behavior at $r = \sqrt{\alpha}$. This exactly parallels recent treatments [9] of euclidean black holes in four dimensions. If we make a change of coordinates $r = \sqrt{\alpha} + \lambda^2$, then the dangerous terms $dr^2/(1-\alpha/r^2) + (1-\alpha/r^2)\, d\phi^2$ become, near $\lambda = 0$, $2\sqrt{\alpha}(d\lambda^2 + (\lambda^2/\alpha)\, d\phi^2)$. This must be compared to the standard expression $d\rho^2 + \rho^2\, d\phi^2$ for the metric of the plane in polar coordinates. We know that $d\rho^2 + \rho^2\, d\phi^2$ describes a non-singular space if (and only if) ϕ is a periodic variable with periodicity 2π. Hence the expression $2\sqrt{\alpha}(d\lambda^2 + (\lambda^2/\alpha)\, d\phi^2)$ describes a non-singular space if and only if ϕ is a periodic variable with periodic $2\pi\sqrt{\alpha}$.

On the other hand, in the Kaluza-Klein vacuum, described by metric (1) or (2), ϕ is a periodic variable with periodicity $2\pi R$. So we must set $\alpha = R^2$ to obtain a non-singular space which asymptotically (for large r) approaches the Kaluza-Klein vacuum. The metric of this space is

$$ds^2 = \frac{dr^2}{1-(R/r)^2} + r^2\, d\Theta^2 + \left(1 - \left(\frac{R}{r}\right)^2\right) d\phi^2 . \tag{4}$$

This space is non-singular and geodesically complete. However, one must note that r is restricted to run from R to ∞, because we had $r = \sqrt{\alpha} + \lambda^2 = R + \lambda^2$, where λ runs from 0 to ∞.

We now wish to see that the solution (4) actually represents an instability of $M^4 \times S^1$. We therefore should look for negative action modes in small fluctuations around (4). Actually, for the analogous solution in one less dimension, this problem has been treated by Perry, who showed that there exists a unique negative action mode. Some implications have been discussed by Gross, Perry, and Yaffe [10].

The generalization of the calculations just mentioned to the metric (4) is discussed in the appendix to this paper. However, there is a much simpler way to see that

(4) describes an instability. This is simply to look for an appropriate analytic continuation of (4) to Minkowski space (that is, to a space of Minkowski signature). According to the general theory of semiclassical vacuum decay, the false vacuum decays into a real Minkowski space solution which agrees with the euclidean bounce solution on a three-dimensional surface which can be regarded as $t=0$. Real euclidean solutions which can be analytically continued to real valued Minkowski solutions (the fields are still real after the continuation) are always found to describe instabilities; this is usually obvious from the form of the Minkowski solution.

An appropriate Minkowski continuation of (4) is easy to find. Letting θ be one of the polar angles, one can write the line element $\mathrm{d}\Theta^2$ of the three sphere as

$$\mathrm{d}\Theta^2 = \mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\Omega^2\,, \tag{5}$$

where $\mathrm{d}\Omega^2$ is the line element of the two sphere.

To continue from euclidean to Minkowski space we should find a plane of symmetry of the metric which can be regarded as $t=0$; then we rotate $t \to it$. In the case at hand, the plane $\theta = \frac{1}{2}\pi$ can play the role of $t=0$ (recall that in flat euclidean space we have $t = r\cos\theta$, if θ is chosen appropriately; see fig. 1). The step $t \to it$ is equivalent to $\theta \to \frac{1}{2}\pi + i\psi$ where ψ is a new real coordinate.

After the replacement $\theta \to \frac{1}{2}\pi + i\psi$ we obtain the Minkowski signature solution of the Einstein equations,

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{(1-(R/r)^2)} - r^2\,\mathrm{d}\psi^2 + \cosh^2\psi\,\mathrm{d}\Omega^2 + (1-(R/r)^2)\,\mathrm{d}\phi^2\,. \tag{6}$$

It is not difficult to check that this space is nonsingular and geodesically complete. The coordinate singularity at $r=R$ is as harmless as it was before the analytic continuation.

The solution (6) is the space into which the Kaluza-Klein vacuum decays. However, what is this space? On this score we will encounter a surprise.

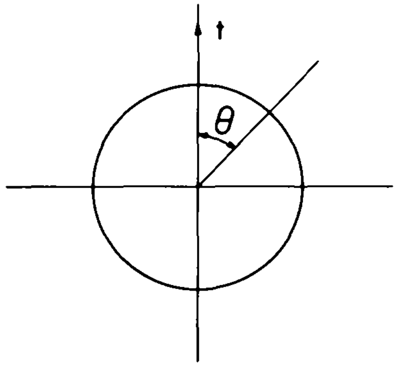

Fig. 1. Time and the polar angle θ in flat euclidean space.

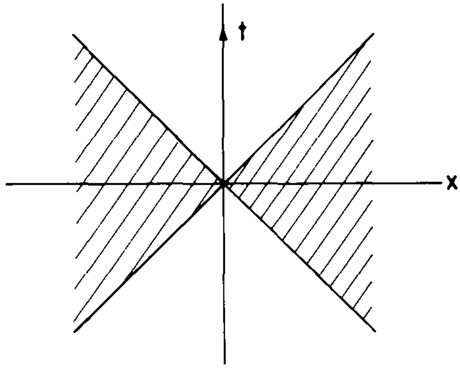

Fig. 2. The exterior of the light cone in Minkowski space.

Let us momentarily drop the factors $(1-(R/r)^2)$ from (6) and consider the two-dimensional space spanned by r and ψ only. So we will examine

$$ds^2 = dr^2 - r^2 \, d\psi^2 . \tag{7}$$

As we might expect, this describes flat two-dimensional Minkowski space. After the change of coordinates

$$x = r \cosh \psi , \qquad t = r \sinh \psi , \tag{8}$$

we find that (7) is converted into

$$ds^2 = dx^2 - dt^2 , \tag{9}$$

which, of course, describes Minkowski space. However, the coordinates r and ψ in eq. (7) do not span all of two-dimensional Minkowski space. From (8) we see that $x^2 - t^2 = r^2$, which is positive for real valued r, so r and ψ only span the exterior of the light cone, $x^2 - t^2 > 0$ (fig. 2). The exterior of the light cone is, of course, not a geodesically complete manifold, because geodesics in Minkowski space can perfectly well reach $x^2 - t^2 = 0$ and continue to negative values of $x^2 - t^2$.

Now we return to the metric of eq. (6). Without the factors of $(1-(r/R)^2)$ this would be the Kaluza–Klein vacuum in an unusual coordinate system, which does not cover the whole space, since it omits the points of negative r^2, where now $r^2 = \boldsymbol{x}^2 - t^2$. The factors of $(1-(R/r)^2)$ are unimportant if r is large, so for large r – large $\boldsymbol{x}^2 - t^2$ – the space of eq. (6) coincides with the Kaluza–Klein vacuum.

What happens when we include the factors $(1-(R/r)^2)$? We know that now r runs not from 0 to ∞ but only from R to ∞. Therefore, if we are to think of (6) as a sort of distorted Minkowski space, it is a space in which not just the interior of the light cone $\boldsymbol{x}^2 - t^2 < 0$ but the interior of a hyperboloid $\boldsymbol{x}^2 - t^2 < R^2$ has been deleted (fig. 3).

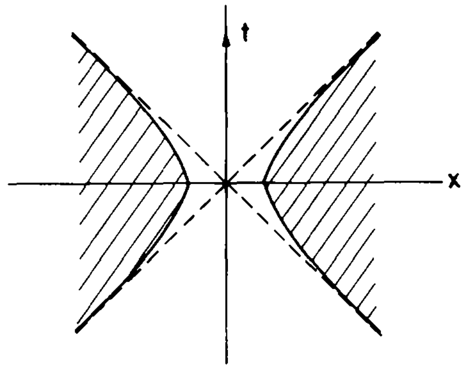

Fig. 3. The exterior of a hyperboloid in Minkowski space.

The Kaluza–Klein space is, of course, a five-dimensional space, with four non-compact dimensions and one compact dimension. However, the spirit of the Kaluza–Klein theory is that the fifth dimension is microscopic and too small to be directly observed.

From the "macroscopic" point of view, for observers who are not able to detect the existence of the fifth dimension and who do not probe too closely near $x^2 - t^2 = R^2$, (6) describes ordinary four-dimensional Minkowski space with the points $x^2 - t^2 < R^2$ omitted. Of course, if one just removes from Minkowski space the region $x^2 - t^2 < R^2$, one obtains not a nice, smooth space but a manifold with a boundary. Here the existence of the fifth dimension is very important. Looking at the term $(1 - (R/r)^2)\,\mathrm{d}\phi^2$ in (6), one sees that the radius of the fifth dimension, while equal to $2\pi R$ asymptotically, is in general $2\pi R\sqrt{1-(R/r)^2}$. The fifth dimension shrinks to zero size as r approaches R in such a way as to smoothly seal off the would-be boundary, giving a non-singular and geodesically complete space.

But looking at (6) as an observer unable to directly see the fifth dimension would look at it, (6) is just Minkowski space with $x^2 - t^2 < R^2$ omitted. It is now easy to describe intuitively what is going on. In the decay of the Kaluza–Klein vacuum into the metric described by (6), a hole spontaneously forms in space. At time zero, when it forms, the hole is microscopic, of radius R. However, the hole rapidly expands. At time t the boundary of the hole is at $x^2 = R^2 + t^2$ so its radius is $r(t) = \sqrt{R^2 + t^2}$. The boundary of the hole is in a state of uniform acceleration, just like the bubble wall in conventional vacuum decay. After a very brief time, of order R, the hole is expanding to infinity at the speed of light.

In a way, this is even more shocking than the conventional decay of a false vacuum. Ordinarily, a false vacuum decays into a more stable state. Here, the Kaluza–Klein vacuum decays – literally – into nothing. A hole spontaneously forms in space and rapidly expands to infinity, pushing to infinity anything it may meet (unless it meets an obstacle massive enough to stop the expansion of the hole).

Actually, the hole will not literally expand to infinity. Because the Kaluza–Klein vacuum is unstable against the formation of holes, holes will form spontaneously at a certain rate per unit volume per unit time. The holes will expand until their boundaries meet. The evolution beyond that point would be difficult to determine, but because of the large energy in the colliding boundaries, one might expect gravitational collapse.

Now let us discuss some lessons that can be learned from this example. One thing that we know about semiclassical vacuum decay is that, like other physical processes, it conserves energy. In conventional false vacuum decay, the energy liberated in the decay to the true vacuum goes into accelerating the bubble wall; the total energy is conserved in this process. Moreover, since no signal travels faster than light, a vacuum decays into another space with the same asymptotic behavior at spatial infinity. Since the Kaluza–Klein vacuum has zero energy, the space (6) into which it decays must, by conservation of energy, also have zero energy. It is easy to check explicitly that this is true. According to the canonical definition (which can be generalized to the Kaluza–Klein theory), the total energy of a system in general relativity is defined as a surface integral in terms of the asymptotic behavior of the gravitational field at spatial infinity. The integral is to be evaluated on an initial value hypersurface, which in (6) can be taken as the surface $\psi = 0$. In the surface integral that defines the energy, only the terms in the metric of order $1/r$ are relevant (in a world of $3+1$ non-compact dimensions). But in (6) there are no terms of order $1/r$; the departure from the flat space metric (coming from the factors $(1-(R/r)^2)$) is of order $1/r^2$. Hence (6) describes a space of zero energy.

This is the difference between Minkowski space and the Kaluza–Klein space $M^4 \times S^1$. Minkowski space has zero energy, and every other solution of Einstein's equations that asymptotically approaches Minkowski space has positive energy. Consequently semiclassical decay of Minkowski space cannot occur (whether matter fields are present or not); the decay would have to produce a space of zero energy that would be asymptotic to Minkowski space, and no such space exists. The key to the instability of the Kaluza–Klein vacuum is that in the Kaluza–Klein case there exist spaces – such as (6) – with the same energy as the Kaluza–Klein vacuum and the same asymptotic behavior.

The statement that every non-flat solution of the Einstein equations that approaches Minkowski space at spatial infinity has positive energy is known as the positive energy theorem. The first full proof was given by Schoen and Yau [11]. A recent review has been given by York [12]. For an alternative proof, and references to the literature, see ref. [7].

What we have discovered, in the explicit counter-example of eq. (6), is that the analogue of the positive energy theorem is not valid for the Kaluza–Klein vacuum. There *are* states other than the vacuum itself which asymptotically approach the vacuum and which have zero energy. Moreover, there are solutions of the Einstein equations with the asymptotic behavior of the Kaluza–Klein vacuum that have *negative* total energy. This can be seen by using methods that were introduced by Brill and Deser [13]. They determined the conditions under which a space can be a stationary point of the energy functional. The space of eq. (6) does not satisfy the Brill–Deser conditions (because the initial value surface $\psi = 0$ is time-symmetric and is not Ricci flat). Hence this space is not a stationary point of the energy functional, and its energy can be raised or lowered by small variations. A variation that makes the energy negative can easily be constructed with the methods of Brill and Deser.

We can now return to a matter that arose at the beginning of this paper. Given an arbitrary Kaluza–Klein type space $M^4 \times B$ which is stable against small fluctuations, how can we assess its stability at the semiclassical level by considerations based on energy?

The answer is not simply to examine the energy of $M^4 \times B$; lacking terms of order $1/r$ in the metric, $M^4 \times B$ automatically has zero energy. The key is to compare the energy of $M^4 \times B$ to the energy of *other states with the same asymptotic behavior.* If the positive energy conjecture holds for $M^4 \times B$, in the sense that every other classical solution with the same asymptotic behavior has positive energy, then $M^4 \times B$ is stable semiclassically.

In general, it may not be easy to decide whether for a given space $M^4 \times B$ the positive energy theorem holds. In some cases, one might hope to answer this question using the methods of ref. [11] or [7], or other methods in the literature. For example, it is interesting to ask why the methods of ref. [7] do not apply to the Kaluza–Klein space.

Because energy is conserved, it is defined not in terms of the whole space-time but as an integral on an initial value surface. For $M^4 \times S^1$ this surface can be taken to be $t = 0$, which defines $R^3 \times S^1$ (R^3 being three-dimensional euclidean space). If one considers excitations of the $M^4 \times S^1$ vacuum which have the same topology as the vacuum, so that the initial value surface is topologically $R^3 \times S^1$ (but is flat only asymptotically) the methods of ref. [7] apply. By considering solutions of the Dirac equation on the initial value surface, one can prove that all excitations of the geometry of $M^4 \times S^1$ in which the topology is not changed have positive energy.

It is therefore not an accident that the space of eq. (6) differs from the Kaluza–Klein ground state in topology and not just in geometry. In fact, the initial value surface $\psi = 0$ in eq. (6) has topology $R^2 \times S^2$, although in its geometry it is asymptotic to the flat metric on $R^3 \times S^1$. The problem that prevents the proof of ref. [6] from applying to the space of eq. (6) is related to this topology. The problem is not that spinors do not exist on $R^2 \times S^2$; they do. Rather, there is a more subtle problem.

Because $R^3 \times S^1$ is not simply connected, there are inequivalent ways to define spinors on this space. One can require the spinors to be periodic functions on S^1 up to a phase $e^{i\alpha}$. The allowed values of the phase are discussed below. In applying the proof of ref. [6] to the Kaluza–Klein theory, one must use spinors with $\alpha = 0$, because only such spinors can be covariantly constant.

$R^2 \times S^2$ is simply connected, so on this space the spinor structure is unique. So the surface $\psi = 0$ in eq. (6) admits a unique spinor structure. This unique spinor structure corresponds, in the asymptotic region where the $\psi = 0$ surface approaches $R^3 \times S^1$, to a unique choice of α on $R^3 \times S^1$. That choice turns out to be $\alpha = \pi$. Since only spinors of $\alpha = 0$ can be used in the proof of ref. [7], and only spinors of $\alpha = \pi$ exist in the space considered here, the proof of ref. [7] does not apply.

In claiming that the bounce solution of eq. (4), and its Minkowski continuation of eq. (6), describe the decay of the Kaluza–Klein vacuum, one must assume that it is appropriate in general relativity to consider manifolds with topology different from the topology of the vacuum. This point should be controversial, because in general relativity, unlike other field theories, cluster decomposition cannot be used to prove the necessity of including varying topologies. In Yang–Mills theory, for instance, magnetic monopoles and instantons must be considered, because a widely separated monopole–antimonopole or instanton–anti-instanton pair can be formed smoothly from the vacuum configuration. Cluster decomposition then forces one to consider isolated monopoles or instantons. Because no analogous argument demonstrates in general relativity the need to include varying topologies, it may be that the interpretation given in this paper is not valid. If, on the contrary, spaces of topology other than the vacuum $M^4 \times S^1$ should not be considered, then the Kaluza–Klein vacuum is stable, for we have noted that the positive energy theorem holds in the Kaluza–Klein theory as long as variations in the topology are not considered.

It should also be noted that in this paper we have considered the "pure" Kaluza–Klein theory, without fields other than the five-dimensional metric. Elementary fermions, if present in the lagrangian, could stabilize the Kaluza–Klein vacuum.

This could arise as follows. We must remember that in defining fermions in the Kaluza–Klein theory, an angle α enters. It enters as follows. One conventionally expands a Fermi field (or other field) in terms of the periodic coordinate ϕ of the fifth dimension as $\psi(x, \phi) = \sum_n \psi_n(x) \exp(in\phi/R)$. With non-zero α, the expansion is instead $\psi(x, \phi) = \sum_n \psi_n(x) \exp(i(n - \alpha/2\pi)\phi/R)$, and the mass squared of the nth state is proportional not to n^2 but to $(n - \alpha/2\pi)^2$.

What values of α are allowed? The basic requirement for an allowed value of α is that the lagrangian of the theory must be invariant under $\psi \to e^{i\alpha}\psi$, so that the lagrangian is single-valued and well-defined even though ψ changes by a phase $e^{i\alpha}$ when ϕ goes from zero to $2\pi R$. The value $\alpha = 0$ is trivially allowed, and for fermions the value $\alpha = \pi$ is also always allowed, because $\psi \to -\psi$ is a symmetry of every Lorentz invariant lagrangian (the total number of fermions is always conserved modulo two). If the lagrangian has additional symmetries, other values of α may be permitted. For instance, if the theory possesses a continuous U(1) law of fermion number conservation, the value of α is completely arbitrary.

Suppose that we construct the Kaluza–Klein vacuum with $\alpha \neq \pi$ (the most obvious choice is $\alpha = 0$). Since in the spaces of eqs. (4) and (6), the definition of spinors is unique and corresponds asymptotically to $\alpha = \pi$, these spaces would not contribute to the same Hilbert space that contains the vacuum. They would contribute to the $\alpha = \pi$ Hilbert space, which has a different spectrum of the elementary particles. Only the vacuum state with $\alpha = \pi$ could decay according to eqs. (4) and (6). One can also see that the fermion determinant with $\alpha = \pi$ vanishes relative to the determinant with $\alpha \neq \pi$ like the exponential of the space-time volume. So in the infinite volume limit, the contribution of (4) to the path integral vanishes relative to $\alpha \neq \pi$ contributions.

Finally, leaving aside elementary fermions, let us ask at what rate the Kaluza-Klein vacuum decays. The action of the five-dimensional theory is

$$I = -\frac{1}{32\pi^2 GR}\int \mathrm{d}^5x\sqrt{g}R + \text{surface term}\,, \tag{10}$$

where a surface term is added to cancel second derivative terms in the action. The gravitational constant in five dimensions is taken to be $2\pi RG$, where R is the radius of the fifth dimension. For the bounce solution, the scalar curvature vanishes, and only the surface term in the action contributes. It can be evaluated to give $\pi R^2/4G$ as the action, so the decay rate of the false vacuum, per unit volume per unit time, is of order $\exp(-\pi R^2/4G)$.

Note that the false vacuum is long-lived if R is much greater than the Planck length. Indeed, only in this case is the semiclassical calculation that we have carried out reliable. If the distances entering are as small as the Planck length, a semiclassical calculation is not reliable quantitatively – although it still strongly indicates an instability.

At the classical level, the radius of the fifth dimension is undetermined. Quantum corrections will give an effective potential that will, in general, determine the radius of the fifth dimension; this effective potential will depend on which matter fields are present. The quantitative validity of the calculation in this paper depends on what the radius of the fifth dimension turns out to be, when quantum effects are included.

For another aspect of Kaluza-Klein dynamics, see ref. [14]. A recent paper on topological aspects of spinors in Kaluza-Klein theory is ref. [15].

I wish to thank D.J. Gross and M. Perry for discussions.

Appendix

In this appendix we will briefly discuss the existence of a negative eigenvalue in the functional determinant obtained in expanding around the euclidean Einstein solution of eq. (4).

As discussed by Perry [5], in a convenient gauge one works with transverse traceless metric fluctuations:

$$g^{\mu\nu}h_{\mu\nu} = D_\mu h^{\mu\nu} = 0\,. \tag{11}$$

The appropriate eigenvalue problem is then

$$\Delta_L h_{\mu\nu} = \lambda h_{\mu\nu} , \tag{12}$$

where Δ_L is the Lichnerowicz laplacian for small fluctuations in the gravitational field.

The most general traceless metric perturbation that preserves the rotational symmetry and time symmetry of eq. (4) is

$$\begin{aligned} h_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = {} & \frac{A(r)}{(1-(r/R)^2)}\, dr^2 + B(r)(1-(r/R)^2)\, d\phi^2 \\ & - \tfrac{1}{3}(A(r)+B(r))r^2\, d\Theta^2 , \end{aligned} \tag{13}$$

where $A(r)$ and $B(r)$ are two arbitrary functions.

As in the four-dimensional problem treated by Perry, B can be expressed in terms of A by using the second part of eq. (11) (transversality). The resulting eigenvalue equation for A [eq. (12)] is a Schrödinger-like equation – a second-order differential equation with an hermitian "hamiltonian".

The existence of a negative eigenvalue of this operator follows from a general argument by Callan and Coleman [4]. If instead of looking at the sector of zero angular momentum, one looks at fluctuations with unit angular momentum, the appropriate ansatz can be obtained from (13) by the simple substitution

$$A(r) \to A(r) \cos\theta , \qquad B(r) \to B(r) \cos\theta , \tag{14}$$

where θ is any one of the polar angles. (For angular momentum bigger than one such a simple substitution would not give the most general ansatz.) Again one obtains an eigenvalue equation for A. In the angular momentum one sector, there exists a zero eigenvalue – it represents the freedom to translate the position of the instanton solution.

At this point the argument of Callan and Coleman can be applied. The eigenvalue problem (12) in the angular momentum one sector differs from the problem in the angular momentum zero sector only in the presence of a strictly positive angular momentum contribution, analogous to $l(l+1)/r^2$ in the Schrödinger equation. The existence of a zero eigenvalue for angular momentum one therefore implies that there is a negative eigenvalue for angular momentum zero.

This agrees with explicit calculation in the four-dimensional problem [5], and with the existence of the unstable minkowskian continuation discussed in this paper.

References

[1] Th. Kaluza, Sitzungsber. Preuss. Akad. Wiss. Berlin, Math. Phys. K1 (1921) 966; O. Klein, Z. Phys. 37 (1926) 895

[2] B. De Witt, *in* Dynamical theory of groups and fields, (Gordon and Breach, NY, 1965);
R. Kerner, Ann. Inst. H. Poincaré 9 (1968) 143;
A. Trautman, Rep. Math. Phys. 1 (1970) 29;
Y.N. Cho, J. Math. Phys. 16 (1975) 2029;
Y.M. Cho and P.G.O. Freund, Phys. Rev. D12 (1975) 1711;
L.N. Chang, K.I. Macrae and F. Mansouri, Phys. Rev. D13 (1976) 235;
J. Scherk and J.H. Schwarz, Phys. Lett. 57B (1975) 463;
E. Cremmer and J. Scherk, Nucl. Phys. B103 (1976) 393; B108 (1976) 409;
R. Casalbuoni, G. Domokos and S. Kovesi-Domokos, Phys. Rev. D17 (1978) 2048

[3] E. Witten, Nucl. Phys. B 186 (1981) 412

[4] S. Coleman, Phys. Rev. D15 (1977) 2929;
S. Coleman and C.G. Callan, Jr., Phys. Rev. D16 (1977) 1762;
P. Frampton, Phys. Rev. Lett. 37 (1976) 1378

[5] M.J. Perry, Instability in quantum gravity, Princeton University preprint, Proc. 1980 Nuffield Workshop, to be published;
D.J. Gross, M.J. Perry and L. Yaffe, Princeton University preprint, to appear

[6] P. Schoen and S.T. Yau, Phys. Rev. Lett. 42 (1979) 547

[7] E. Witten, Comm. Math. Phys. 80 (1981) 381

[8] F.R. Tangherlini, Nuovo Cim. 27 (1963) 636

[9] G.W. Gibbons and M.J. Perry, Proc. Roy. Soc. A356 (1978) 467

[10] D.J. Gross, M.J. Perry and L. Yaffe, ref. [5]

[11] P. Schoen and S.T. Yau, Comm. Math. Phys. 65 (1979) 45; Phys. Rev. Lett. 43 (1979) 1457; Comm. Math. Phys., to be published

[12] J.W. York, Jr., *in* Essays on general relativity, ed. Frank Tipler (Academic Press, 1980)

[13] D. Brill and S. Deser, Ann. of Phys. 50 (1968) 548

[14] A. Chodos and S. Detweiler, Phys. Rev. D21 (1980) 2167

[15] W. Mecklenberg, ICTP preprint (March, 1981)