"Irreversibility" of the flux of the renormalization group in a 2D field theory

A. B. Zomolodchikov
L. D. Landau Institute of Theoretical Physics, Academy of Sciences of the USSR

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There exists a function $c(g)$ of the coupling constant g in a 2D renormalizable field theory which decreases monotonically under the influence of a renormalization-group transformation. This function has constant values only at fixed points, where c is the same as the central charge of a Virasoro algebra of the corresponding conformal field theory.

The renormalization group is one of the most powerful methods for qualitative studies in field theory.[1,2] The procedure for determining the renormalization group can be roughly summarized as follows[1]: We denote by $S(g,a)$ an action functional of a (Euclidean) field theory which is an integral of the local density, $S=\int\sigma(g,a,x)dx$, equipped with an ultraviolet cutoff a and depending on a (possibly infinite) set of dimensionless parameters $g=(g^1,g^2,\ldots)$, which are known as "coupling constants." A basic assumption is that there exists a single-parameter group of motions in the space (Q) of coupling constants g, $R_t Q\rightarrow Q$, of such a nature that a field theory describable by an action $S(R_t g, e^t a)$ is equivalent to the original theory with the action $s(g,a)$ in the sense that all the correlation functions of the two theories agree at scales $x\gg e^t a\,(t>0)$. The components of the vector field which generate the renormalization group are called "β functions":

$$dg^i = \beta^i(g)dt. \tag{1}$$

Some of the information on the ultraviolet behavior of the field theory is lost under renormalization transformations with $t>0$, since in the field theory it is not legitimate to examine correlations at scales smaller than the cutoff. We would therefore expect that a motion of the space Q under the influence of the renormalization group would become an "irreversible" process, similar to the time evolution of dissipative systems. In the present letter we restrict the discussion to a 2D field theory, and for this case we establish the following general properties of the renormalization group:

1. There exists a function $c(g)\geqslant 0$ of such a nature that we have

$$\frac{d}{dt}c \equiv \beta^i(g)\frac{\partial}{\partial g^i}c(g) \leqslant 0 \tag{2}$$

(a repeated index implies a summation). The equality in (2) is reached only at fixed points of the renormalization group, i.e., at $g=g_*$ $[\beta^i(g_*)=0]$.

2. The fixed points (here and below, we mean "critical" fixed points, at which the correlation radius is infinite[1]) are stationary for $c(g)$; i.e., we have $\beta^i(g)=0\rightarrow\partial c/\partial g^i=0$. At the critical fixed points, the 2D field theory has an infinite conformal symmetry.[3] The corresponding generators L_n, $n=0,\ \pm 1,\ \pm 2,\ \ldots$, form a Virasora algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{\tilde{c}}{12}(n^3-n)\delta_{n+m,0}, \tag{3}$$

where the numerical parameter $\tilde{c}$ (the "central charge") is an important characteristic of a conformal field theory.[3,4] It generally takes on different values for different fixed points; i.e., $\tilde{c}=\tilde{c}(g_*)$.

3. The value of $c(g)$ at the fixed point g_* is the same as the corresponding central charge in (3); i.e., $c(g_*)=\tilde{c}(g_*)$.

The proof of this "c-theorem" is based on the conditions of renormalizability, positivity,[5] the translational and rotational symmetries of the field theory, and certain special properties of a 2D conformal field theory. Spatial symmetries in a local field theory lead to the existence of a local energy-momentum tensor $T_{\mu\nu}(x)=T_{\nu\mu}(x)$ which satisfies the equation $\partial_\mu T_{\mu\nu}=0$. We introduce the complex coordinates $(z,\bar{z})=(x^1+ix^2, x^1-ix^2)$, and we use the notation $T=T_{zz}$, and $\Theta=T_{z\bar{z}}$. We also define the scalar local fields

$$\Phi_i(g,x) = \frac{\partial}{\partial g^i}\sigma(g,a,x). \tag{4}$$

The exact meaning of the assertion that a field theory is renormalizable is that for all g the field Θ can be expanded in basis (4):

$$\Theta = \beta^i(g)\Phi_i, \tag{5}$$

where the coefficients $\beta^i(g)$ are the same as in (1). We define the functions

$$C(g) = 2z^4\langle T(x)T(0)\rangle|_{x^2=x_0^2}; \tag{6a}$$

$$H_i(g) = z^2x^2\langle T(x)\Phi_i(0)\rangle|_{x^2=x_0^2}; \tag{6b}$$

$$G_{ij}(g) = x^4\langle \Phi_i(x)\Phi_j(0)\rangle|_{x^2=x_0^2} \tag{6c}$$

where $x_0\gg a$ is an arbitrary scale ("normalization point"). At this point we set $x_0=1$. By virtue of the positivity condition in the field theory,[5] the symmetric matrix $G_{ij}(g)$ is positive definite and may be thought of as the metric in Q. Combining the requirement $\partial_\mu T_{\mu\nu}=0$ with (5,) and with the Callan-Simanchik equation,[2] we find the relations

$$\frac{1}{2}\beta^i\partial_i C = -3\beta^i H_i + \beta^i\beta^k\partial_k H_i + \beta^k(\partial_k\beta^i)H_i; \tag{7a}$$

$$\beta^k\partial_k H_i + (\partial_i\beta^k)H_k - H_i = -2\beta^k G_{ik} + \beta^j\beta^k\partial_k G_{ij} + \beta^j(\partial_i\beta^k)G_{jk} + \beta^j(\partial_j\beta^k)G_{ik}. \tag{7b}$$

where $\partial_i=\partial/\partial g^i$. In deriving (7) we made use of the following expression for the

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matrix $\gamma_i^j(g)$:

$$\gamma_i^j(g)\,\Phi_j \equiv \left(\frac{1}{2}\,a\frac{\partial}{\partial a} - \beta^k\frac{\partial}{\partial g^k}\right)\Phi_i = (\partial_i\beta^j)\Phi_j \qquad (8)$$

For the function

$$c(g) = C(g) + 4\beta^i H_i - 6\beta^i\beta^j G_{ij} \qquad (9)$$

we find from (7)

$$\beta^i\partial_i c = -12\beta^i\beta^j G_{ij}\,, \qquad (10)$$

directly verifying Assertion 1. Assertion 3 follows from (9) and from the definition of the central charge $\tilde{c}(g_*)$ as the numerical coefficient in the correlation function[3] $\langle T(z)T(0)\rangle_{g_*} = z^{-4}\tilde{c}(g_*)/2$. To prove Assertion 2, we consider the critical fixed point g_*, and we choose a coordinate system in Q such that we have $g_* = 0$ and

$$G_{ij}(g) = \delta_{ij} + 0(g^2)\,. \qquad (11)$$

In this case the vectors $\Phi_i(g_*,x)$ are conformal fields and have certain anomalous dimensionalities d_i. Near the point $g_* = 0$, the function $c(g)$ can be calculated by perturbation theory; the result is

$$c(g) = \tilde{c}(g_*) - 6\epsilon_i g^i g^i + 2C_{ijk}\,g^i g^j g^k + O(g^4)\,, \qquad (12)$$

where $2\varepsilon_i = 2 - d_i$. Assertion 2, in particular, follows from (12). We also note that in the special case of "soft" perturbations, with $|\epsilon_i| \ll 1$, it can be shown that the coefficients C_{ijk} in (12) are the same as the structure constants of the operator algebra of a conformal field theory,[3] g_*. For the β functions we find

$$\beta^i(g) = \epsilon_i g^i - \frac{1}{2}C_{ijk}\,g^j g^k + O(g^3)\,. \qquad (13)$$

(No summation is to be carried out in the first term.) At the specified accuracy, therefore, the following relation holds near the fixed point:

$$\beta^i(g) = -\frac{1}{12}\,G^{ij}(g)\,\frac{\partial}{\partial g^j}c(g)\,, \qquad (14)$$

where $G^{ik}G_{kj} = \delta^i_j$.

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Asymmetry in the reaction $d(e,e'd)$ at a momentum transfer of 1–1.5 fm^{-1}

B. B. Voĭtsekhovskiĭ, D. M. Nikolenko, K. T. Ospanov, S. G. Popov, I. A. Rachek, D. K. Toporkov, E. P. Tsentalovich, and Yu. M. Shatunov
Institute of Nuclear Physics, Siberian Branch, Academy of Sciences of the USSR

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The asymmetry in the cross section for the reaction $d(e,e'd)$ has been studied at $E_e = 400$ MeV and $\theta_e = 30$–$50°$ with a tensor-polarized jet target of atomic deuterium in the VÉPP-2 electron storage ring. An analyzing power $F_{20} = 0.18 \pm 0.07$ is found. This power determines the ratio of the quadrupole and monopole electric form factors of the deuteron.

Polarization experiments with deuterium can resolve several questions of current interest in the physics of nucleon-nucleon interactions.[1] The experiment which we are reporting in the present letter is a continuation of a study[2] being carried out to separately measure the monopole and quadrupole electric form factors of the deuteron.

We write the reaction cross section at $q < 2$ fm^{-1} as

$$d\sigma/d\Omega_e = (d\sigma_0/d\Omega_e)\left\{1 - \frac{1}{\sqrt{2}}F_{20}P_{zz}P_2\,(\cos(\mathbf{h}\mathbf{q}/|\mathbf{q}|))\right\},$$

where $d\sigma_0/d\Omega_e$ is the scattering cross section of the unpolarized deuteron, F_{20} is the analyzing power, P_{zz} is the degree of tensor polarization, **h** is a unit vector along the polarization direction, and **q** is the momentum transfer. At small values of the momentum transfer, F_{20} is equal to

$$\sqrt{\tfrac{2}{9}}\,\frac{q^2}{M_d^2}G_Q/G_E$$

where G_E and G_Q are the monopole and quadrupole form factors of the deuteron, and M_d is the mass of the deuteron.

By measuring the reaction cross section for various angles between **h** and **q** and also for various values of P_{zz}' we can determine F_{20}.

The experimental arrangement is shown in Fig. 1. An electron beam (average current of 0.25 A, diameter of 3–4 mm) intersects a jet of deuteron atoms[3] (density $\sim 10^{11}$ atoms/cm^3, diameter of 7 mm). The polarization direction of the deuterons in the region in which the electron beam intersects the jet is determined by the magnetic field, whose vector lies in the reaction plane (for the case $\varphi = 0°$), making an angle $\theta = 44°$ or $132°$ with the beam axis (the cases H_1 or H_2 in Fig. 1). The degree of polarization of the atoms in the jet is determined from the distribution of their deflec-