

DYONS OF CHARGE $e\theta/2\pi$ E. WITTEN¹

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It is shown that in CP non-conserving theories, the electric charge of an 't Hooft–Polyakov magnetic monopole will not ordinarily be integral, or even rational in units of the fundamental charge e . If a non-zero vacuum angle θ is the only mechanism for CP violation, the electric charge of the monopole is exactly calculable and is $-e\theta/2\pi$, plus an integer. If there are additional CP violating interactions, the monopole charge must be computed as a power series in the coupling constant. These results apply in realistic theories such as $SU(5)$.

Long ago, Dirac showed [1] that the quantum mechanics of an electrically charged particle of charge e and a magnetically charged particle of charge g is consistent only if $eg = 2\pi n$, n being an integer.

Zwanziger [2] and Schwinger [3] generalized this condition to allow for the possibility of particles (dyons) that carry both electric and magnetic charge. A quantum mechanical theory can have two particles of electric and magnetic charges (e_1, g_1) and (e_2, g_2) only if $e_1 g_2 - e_2 g_1 = 2\pi n$.

This formula may be heuristically derived [4] by considering the classical formula for the angular momentum of the electromagnetic field. The angular momentum in the field of the two particle system can be calculated readily. It has magnitude $(e_1 g_2 - e_2 g_1)/4\pi c$. This has an integer or half-integer value, as expected in quantum mechanics, only if $(e_1 g_2 - e_2 g_1)/\hbar c = 2\pi n$.

Since in nature there are electrons of charges $(e, 0)$, the quantization condition, applied to a hypothetical magnetic monopole of charges (q, g) , requires $eg = 2\pi n$. Notice that, because the electron has no magnetic charge, the electric charge of the monopole does not contribute to $e_1 g_2 - e_2 g_1$. Therefore, the quantization condition, by itself, says nothing about the elec-

tric charge that a magnetic monopole should be expected to have.

The quantization condition does say something about the difference between the electric charges of two magnetic monopoles. Given, for instance, two monopoles of minimum allowed magnetic charge $g = 2\pi/e$ and of electric charges q and q' , one finds $e_1 g_2 - e_2 g_1 = 2\pi(q - q')/e$, so that the Dirac–Schwinger–Zwanziger condition gives

$$q - q' = ne. \quad (1)$$

Thus, the difference $q - q'$ must be an integral multiple of e . But, as Zwanziger and Schwinger noted, there is no restriction on q and q' separately.

If, however, the Dirac quantization condition is supplemented by CP conservation, the allowed values of the electric charge of a magnetic monopole are quantized. In fact, although the electric charge is odd under CP , the magnetic charge is even. (This is so because electric and magnetic fields transform oppositely under parity.) Applied to a monopole of charges $(q, 2\pi/e)$, a CP transformation gives a monopole of charges $(-q, 2\pi/e)$. For these two particles, $e_1 g_2 - e_2 g_1 = 4\pi q/e$, and is a multiple of 2π only if

$$q = ne, \quad \text{or} \quad q = (n + \frac{1}{2})e. \quad (2)$$

Thus, the monopoles must have integer or half-integer charges. Moreover, in view of eq. (1), if monopoles of integer charge exist, then monopoles of half-integer

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charge do not, and vice-versa.

Apart from *CP* conservation, there is no general reason to expect magnetic monopoles to have integral (or even rational) electric charges. In nature, *CP* is violated, but only weakly. One may therefore suspect that monopoles, if they exist, have charges that are almost, but not quite, integers. The deviation of the monopole from integral charge would be proportional to the strength of *CP* violation. The purpose of this paper is to discuss this question within the context of current-day gauge theories.

Gauge theories in which electromagnetism arises from the spontaneous breakdown of a compact gauge symmetry are known to have magnetic monopoles, with values of the magnetic charge that satisfy the Dirac condition. This was originally discovered by 't Hooft [5] and Polyakov [6], who found solutions of the classical field equations with non-zero magnetic charge. (For some reviews, see ref. [7].)

The classical field equations also have dyon solutions [8], that is, solutions describing configurations of both electric and magnetic charge. After quantization, these become quantum states carrying both types of charge.

To determine the electric charges carried by the dyons at the quantum level is somewhat delicate. At the classical level, the dyon charge is completely unrestricted—classical solutions exist for any value of the dyon charge.

The question of the charge of the quantum dyons has been studied carefully by semi-classical reasoning [9] and it has been concluded that the dyon charge is quantized to be an integer multiple of the fundamental charge, $q = ne$. These analyses, however, have been carried out in *CP* conserving theories; here we will consider the consequences of *CP* violation.

Although the arguments that follow are general, and apply also to “realistic” theories such as the $SU(5)$ grand unified theory [10], it is useful to consider the simplest theory that has magnetic monopoles. This is the theory of an $O(3)$ gauge group, spontaneously broken to $U(1)$ by the vacuum expectation value of an isovector field ϕ :

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} D_\mu \phi^2 - \lambda(\phi^2 - a^2)^2. \quad (3)$$

An interesting way to introduce *CP* violation into this theory is to consider a non-zero value of the recently discovered [11] vacuum angle θ . One adds to the

lagrangian an additional, *CP* violating interaction

$$\Delta\mathcal{L} = \theta (e^2/32\pi^2) F_{\mu\nu} \cdot \tilde{F}_{\mu\nu}. \quad (4)$$

As has been discovered in the last few years, this additional interaction, despite being superficially a total divergence, modifies the physics. When θ is not zero, *CP* is not conserved.

As will become clear, there is a rather close connection between θ and the electric charge of the dyon. The effect of θ on the dyon charge will be explicitly computed below. The effects of other forms of *CP* violation will be discussed qualitatively.

To determine the effect of *CP* violation on the dyon charge, one must repeat the existing semi-classical analysis of the dyon charge in the presence of *CP* violation. A particularly simple approach is to follow the reasoning of Tomboulis and Woo [12]. They described a semi-classical quantization of the classical dyon solutions. In a gauge in which the gauge field vanishes at infinity, the classical dyon solution is periodic in time. The semi-classical quantization condition is that $S + ET$, the action in a period plus the energy times the time, should be a multiple of 2π .

The action S of the dyon solution in a period is equal to the period T times the action per unit time I , so the requirement is $T(I + E) = 2\pi n$. The classical period T and the “abbreviated action” $I + E$ were calculated in the absence of *CP* violation by Tomboulis and Woo, who found

$$I + E = cq^2, \quad (5)$$

$$T = (2\pi/ec)(1/q), \quad (6)$$

where q is the charge of the dyon, and c is a certain constant. (It is not easy to calculate c , but a simple argument shows that the same constant c appears in eqs. (5) and (6).) The condition $T(I + E) = 2\pi n$ now gives simply

$$q = ne, \quad (7)$$

so that the dyons have integral charges, as one might expect in the absence of *CP* violation.

Let us now repeat this calculation at non-zero θ . At non-zero θ the equations of motion are unchanged, and there is no change in the period T or energy E . However, there is an extra contribution to the action I from the extra term (4) in the lagrangian. The extra term can be readily evaluated, and one finds that

at non-zero θ

$$I + E = cq^2 + ceq\theta/2\pi, \tag{8}$$

with the same constant c as before. Semi-classical quantization of $T(I + E)$ now gives

$$q = ne - \theta e/2\pi, \tag{9}$$

so the allowed values of the magnetic monopoles electric charge depend on θ and are not integral if θ is not zero. In particular, if θ is not zero, there does not exist an electrically neutral magnetic monopole.

It may come as a surprise that the θ dependence can be calculated in this way, without mentioning instantons. In the absence of magnetic monopoles, there are, in this theory, no classically allowed motions with non-zero $\int d^4x F_{\mu\nu} \cdot \tilde{F}_{\mu\nu}$. The θ dependence arises therefore as a tunnelling effect, connected with instantons, and is of order $\exp(-1/\alpha)$. (Because of the Higgs field, the instantons have a natural size, and there are no divergent scale integrations.) But in the monopole sector, there are classically allowed motions—dyons—with non-zero $\int d^4x F_{\mu\nu} \cdot \tilde{F}_{\mu\nu}$. (This aspect of the monopole has been stressed before by Pagels and Marciano [13].) As a result, the θ dependence in the monopole sector has nothing to do with instantons, and is of leading order rather than order $e^{-1/\alpha}$.

What happens in theories in which CP is violated by some mechanism other than θ ?

The fact that at $\theta = 0$ the dyons have integer charges is related, as the above derivation shows, to the fact that $I + E$ in eq. (5) is quadratic in q , with no linear term. A linear term, as in eq. (8), leads to non-integral charges, the non-integrality being proportional to the coefficient of the linear term.

CP forbids a term linear in q because q is odd under CP . If CP is violated, regardless of the mechanism, a linear term may be present.

Even if the linear term is absent at the classical level (this happens, for instance, if only the couplings to fermions violate CP , since the fermions do not enter in the classical solutions), loop corrections can still generate an effective linear term. Roughly speaking, one should recalculate $I + E$ from the quantum effective action rather than the classical action. If CP is violated, loop correction to the effective action should be expected to induce a term linear in q in the effective $I + E$ and therefore to cause the monopole charges

to be not quite integers.

In general, the monopole charges have the form $(n + \delta)e$, where δ , the deviation from integrality, can only be computed as a power series in the coupling constant. But remarkably, if θ is the only source of CP violation, formula (9), $\delta = -\theta/2\pi$, is exact with no higher order corrections.

This can be established by canonical reasoning. Let us attempt to define the operator N that generates gauge transformations around the direction $\hat{\phi}$ —these transformations would ordinarily correspond to electric charges. N will generate the transformation $\delta\mathbf{v} = (1/a)\hat{\phi} \times \mathbf{v}$ for any isovector field \mathbf{v} , and $\delta A_\mu = (1/ea) \times D_\mu \hat{\phi}$ for the gauge field (a is the vacuum expectation value of ϕ).

The eigenvalues of the operator N are integers; indeed, as an operator statement, $e^{2\pi iN} = 1$. This statement usually corresponds to quantization of electric charge; here we will see it corresponds to quantization of a certain linear combination of electric and magnetic charge. The reason that $e^{2\pi iN}$ equals one is that it generates a 2π rotation around $\hat{\phi}$. A 2π rotation is no transformation at all, and so leaves the states invariant, giving $e^{2\pi iN} = 1$.

More accurately, only at spatial infinity, where ϕ has magnitude a , does $e^{2\pi iN}$ generate a 2π rotation around $\hat{\phi}$. Elsewhere the rotation angle is $2\pi|\phi|/a$. But for gauge invariant physical states the action of a gauge transformation depends only on the behaviour of the transformation at infinity; if it equals one at infinity, it leaves the states invariant. (In the formulation used here, in which θ is regarded as a coupling constant, this is true even for a topologically non-trivial gauge transformation.) So $e^{2\pi iN} = 1$.

Now let us compute N . We use Noether's formula,

$$N = \frac{\delta \mathcal{L}}{\delta \partial_0 A_\mu} \cdot \delta A_\mu + \frac{\delta \mathcal{L}}{\delta \partial_0 \phi} \cdot \delta \phi. \tag{10}$$

With $\delta A_\mu = (1/ea) D_\mu \hat{\phi}$ and $\delta \phi = 0$, and remembering to include the contribution from the $\theta F\tilde{F}$ term, one finds

$$N = \frac{1}{ea} \int d^3x D_i \hat{\phi} \cdot F_{0i} + \frac{\theta e}{8\pi^2 a} \int d^3x D_i \hat{\phi} \cdot (\frac{1}{2} \epsilon_{ijk} F_{jk}). \tag{11}$$

In this theory a conventional definition of the electric and magnetic charge operators Q and M would be

$$Q = \frac{1}{a} \int d^3x D_i \phi \cdot F_{0i} = \frac{1}{a} \int d^3x \partial_i (\phi \cdot F_{0i}),$$

$$M = \frac{1}{a} \int d^3x D_i \phi \cdot (\frac{1}{2} \epsilon_{ijk} F_{jk}),$$

$$= \frac{1}{a} \int d^3x \partial_i (\phi \cdot \frac{1}{2} \epsilon_{ijk} F_{jk}),$$
(12)

where the equations of motion have been used to relate the middle and last terms of each equation (recall that $(1/a)\phi \cdot F_{0i}$ is the gauge invariant electric field, whose divergence is the electric charge density). So we have

$$N = ((1/e)Q + (\theta e/8\pi^2)M) = 1 \quad (13)$$

Since $e^{2\pi i N} = 1$, we obtain an operator statement

$$\exp [2\pi i ((1/e)Q + (\theta e/8\pi^2)M)] = 1 \quad (14)$$

and it is $(1/e)Q + (\theta e/8\pi^2)M$ that has integer eigenvalues.

Applied to the 't Hooft–Polyakov magnetic monopole, which has $M = 4\pi/e$ (twice the Dirac value, because in this theory one could introduce isospinor particles of charge $e/2$), this leads us back to our previous conclusion that the allowed eigenvalues of Q satisfy $q = ne - e\theta/2\pi$.

Some readers might wonder whether the same result is obtained if one regards θ not as a coupling constant multiplying a certain term in the lagrangian but as the phase that the states acquire under a topologically non-trivial gauge transformation. In this case, because there is no $\theta \tilde{F}\tilde{F}$ term in the lagrangian N is simply Q/e . However, it is no longer true that $\exp(2\pi i N) = 1$. In the magnetic monopole sector, $\exp(2\pi i N)$ generates a gauge transformation which is topologically non-trivial, with winding number -1 . This is related to the way that ϕ in the Higgs sector is topologically twisted. In the formalism in which θ resides in the states rather than in the lagrangian, the magnetic monopole states transform under $\exp(2\pi i N)$ like $e^{-i\theta}$. With $N = Q/e$, we have $\exp(2\pi i Q/e) = e^{-i\theta}$ in the magnetic monopole sector, leading again to the allowed values of electric charge $q = ne - \theta e/2\pi$.

Notice that in the magnetic monopole sector the “topologically non-trivial” gauge transformations do

not really deserve that name. For $\exp(2\pi i N)$ can be reached continuously from the identity as $\exp(i\alpha N)$, with α varied from 0 to 2π , and N is an “allowed” gauge transformation that does not change the values of the fields at infinity. In the vacuum sector $\exp(2\pi i N)$ has zero winding number, and there is no way to obtain a gauge transformation of non-zero winding number as the exponential of an allowed gauge transformation. The fact that the “topologically non-trivial” gauge transformations can be reached continuously from the identity in the magnetic monopole sector is another way to understand the fact that in this sector θ , rather than being a tunnelling effect, is present in the leading semi-classical approximation.

Eq. (14) is a symmetry statement and shows that the formula $q = ne - \theta e/2\pi$ is exact.

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