Nuclear Physics B262 (1985) 263-283

Q-BALLS*

Sidney COLEMAN

Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138, USA

Received 4 July 1985

A large family of field theories in 3+1 dimensions contains a new class of extended objects. The existence of these objects depends on (among other conditions) the existence of a conserved charge, Q, associated with an ungauged unbroken continuous internal symmetry. These objects are spherically symmetric, and for large Q their energies and volumes grow linearly with Q; thus they act like homogeneous balls of ordinary matter, with Q playing the role of particle number. This paper proves the fundamental existence theorem for these Q-balls, computes their elementary properties, and finds their low-lying excitations.

1. Introduction and conclusions

This paper is an introduction to a new class of extended objects (sometimes called lumps or solitons) arising in a large family of field theories in four space-time dimensions with unbroken continuous global symmetries.

The simplest theory displaying the phenomenon is the SO(2)-invariant theory of two real scalar fields with nonderivative interactions. The theory is defined by the Lagrange density**

$$\mathscr{L}=\tfrac{1}{2}(\partial_\mu\phi_1)^2+\tfrac{1}{2}(\partial_\mu\phi_2)^2-U(\phi)\,, \tag{1.1}$$

where $\phi=\sqrt{\phi_1^2+\phi_2^2}$. The SO(2) symmetry is

$$\begin{aligned}\phi_1&\to\phi_1\cos\alpha-\phi_2\sin\alpha\,,\\ \phi_2&\to\phi_1\sin\alpha+\phi_2\cos\alpha\,.\end{aligned} \tag{1.2}$$

The associated conserved current is

$$j_\mu=\phi_1\partial_\mu\phi_2-\phi_2\partial_\mu\phi_1\,, \tag{1.3}$$

and the conserved charge is

$$Q=\int \mathrm{d}^3\boldsymbol{x}\, j_0\,. \tag{1.4}$$

* Research supported in part by the National Science Foundation under Grant No. PHY82-15249.

** Notation: Greek indices run from 0 to 3. Latin indices run from 1 to 3. Boldface indicates three-vectors (and sometimes internal-space two vectors). The signature of the space-time metric is $(+---)$. $\dot\phi$ is $\partial\phi/\partial t$. U' is $\mathrm{d}U/\mathrm{d}\phi$.

In this paper, I will restrict myself to the simple theory defined by eq. (1.1). However, most of the analysis can be extended in a straightforward way to more complicated theories with larger internal symmetry groups. Also, I will treat the theory in the leading semiclassical approximation only; that is to say, I will do classical field theory, occasionally disguised by quantum language.

By convention, $U(0)=0$. If this is the absolute minimum of U (the case of interest), $\phi=0$ is the ground state of the theory and the SO(2) symmetry is unbroken. The perturbative particle spectrum consists of spinless mesons with $Q=\pm 1$ and mass μ, where

$$\mu^2 = U''(0) = [2U/\phi^2]_{\phi=0}\,. \tag{1.5}$$

In the body of this paper, I shall show that:

(i) New particles appear in the spectrum of the theory if U is such that the minimum of U/ϕ^2 is at some point $\phi_0 \neq 0$. In equations,

$$\min\,[2U/\phi^2] \equiv 2U_0/\phi_0^2 < \mu^2\,. \tag{1.6}$$

To be more precise, in this case, for sufficiently large Q, there exist nondissipative solutions of the classical field equations that are absolute minima of the energy for fixed Q. Thus they are absolutely stable, not just stable under small deformations.

(ii) For appropriate choice of the origin of space-time, these solutions are of the form

$$\begin{aligned}\phi_1 &= \phi(r)\cos\omega t\,,\\ \phi_2 &= \phi(r)\sin\omega t\,,\end{aligned} \tag{1.7}$$

where $\phi(r)$ is a monotonically decreasing function of distance from the origin, going to zero at infinity, and ω is a constant. In other words, these objects rotate with constant angular velocity in internal space and are spherically symmetric in position space.

(iii) As Q goes to infinity, ω approaches

$$\omega_0 = \sqrt{2U_0/\phi_0^2}\,. \tag{1.8}$$

In this same limit, ϕ resembles a smoothed-out step function. For r less than a certain radius, R, $\phi=\phi_0$; outside this radius, $\phi=0$; these two regions are connected by a transition zone with thickness on the order of μ^{-1}. R can easily be computed from eq. (1.4), which implies

$$Q = \tfrac{4}{3}\pi R^3 \omega_0 \phi_0^2\,. \tag{1.9}$$

This is very much like the description of a ball of ordinary matter, some substance that has a thermodynamic limit; the values of local quantities inside a sample are independent of sample size for sufficiently large samples. The stability of ordinary matter depends on the conservation of particle number, and the radius of a ball of ordinary matter depends on the number of particles in it. Here, the role of particle number is played by Q. For this reason I call these systems "Q-balls"; the homogeneous state that exists in their interiors I call "Q-matter".

(iv) A sphere of ordinary matter has a rich spectrum of small vibrations about its equilibrium state. Some of these (sound waves and surface waves) have minimum frequencies that to to zero as the radius of the sphere goes to infinity. These lead in quantum theory to extremely low-lying excited states. The same is true for Q-matter.

The body of this paper is devoted to establishing these results. Sect. 2 derives the essential properties of Q-balls, except for the proof of absolute stability. Sect. 3 is the proof. It is long and dull, and can easily be skipped by a reader blind to the beauties of real analysis. Sect. 4 is a computation of the small-vibration spectrum.

Before plunging into this analysis, though, I would like to make some comments:

(i) Some years ago, various subsets of Friedberg, Lee, Sirlin, and Wick wrote a sequence of papers exploring what they called nontopological solitons [1]. Q-balls in some ways resemble these objects; for example, in both cases, stability depends on the existence of an unbroken continuous symmetry with its associated conservation law. However, in other ways they are very different; for example, the constructions of Friedberg et al. all depend on the existence of a second symmetry which is spontaneously broken; this has no analog for Q-balls. I think it fair to say that these two kinds of objects are in the same family but not the same genus.*

(iii) The only renormalizable interaction allowed for our theory is $U = \frac{1}{2}\mu^2\phi^2 + \lambda\phi^4$, with both μ^2 and λ positive. Such a function never satisfies eq. (1.6). This is not a problem, for two reasons. Firstly, U need not be a fundamental interaction; it could be an effective interaction, obtained by integrating over other degrees of freedom, or by taking account of thermal effects. The difference between such an effective interaction and a fundamental interaction would be important were we to compute one-loop effects, but it is irrelevant as long as we restrict ourselves to the leading approximation. Secondly, as soon as we complicate the theory slightly, we can create renormalizable interactions that obey the appropriate analog of eq. (1.6). For example, if we put ϕ in the adjoint representation of SU(3), we can add a term proportional to $\mathrm{Tr}\,\phi^3$ to U, and, for appropriate choice of the coefficient, this is sufficient to insure the existence of Q-matter [2].

(iii) Q-balls occur whenever we are near (but not at) a first-order symmetry-breaking phase transition. Fig. 1 is a sketch of U for such a situation. U_+, the value of U at the local minimum, ϕ_+, is positive, so the symmetry is unbroken; however, if U_+ is sufficiently small, $2U_+/\phi_+^2$ is less than μ^2, so Q-balls exist. It is instructive to work out what happens if we adiabatically alter the parameters of the theory to approach the phase transition. As U_+ goes to zero, ϕ_0 goes to ϕ_+, ω_0 goes to zero, and R (for fixed Q) goes to infinity. That is to say, the Q-ball adiabatically becomes the new (asymmetric) vacuum.

(iv) This suggests an alternative scenario for a first-order phase transition in cosmic evolution. Instead of the nuclei of the new phase appearing after the critical

* *Note added in proof*: There is more to be said on this. See the appendix.

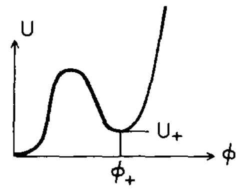

Fig 1

Fig. 1.

temperature is reached, as a result of vacuum tunneling, they might appear before, in the form of Q-balls. Presumably, once Q-balls become possible, they will inevitably condense from Q-density fluctuations in a pre-existing meson plasma. (Remember, there's no long-range force to suppress such fluctuations.) Unfortunately, I have no idea how to estimate the rate of this process. In any event, this alternative does nothing for the problems of the old inflationary cosmology; these arise because the old phase expands faster than the bubbles of new phase grow, not because of any difficulty in forming bubble nuclei initially [3].

(v) I have been unable to construct Q-balls when the continuous symmetry is gauged. I think what is happening physically is that the long-range force caused by the gauge field forces the charge inside the Q-ball to migrate to the surface, and this destabilizes the system, but I am not sure of this. (Of course, this is just what happens to nuclear matter in the real world.)

(vi) Can Q-balls exist now? The obvious candidate for the relevant ungauged unbroken continuous symmetry is baryon number. I know of no reason why the vacuum energy density, as a function of some (possibly composite) baryon-number violating field, might not have a secondary minimum far from the origin; if this were so, we would have B-balls. Of course, if baryon number is not strictly conserved (as grand unification suggests), the B-balls would be unstable, but this is not necessarily a serious problem; they would still be interesting, even if they were as evanescent as protons.

2. Building Q-balls

I will begin with a quick and dirty computation. I will assume that there exists a solution to the equations of motion of the general form discussed in the Introduction: within some sphere of volume V, ϕ is a constant; outside the sphere, it is zero. Furthermore, ϕ is in steady rotation in internal space, with some frequency ω, as in eq. (1.7). I will attempt to find the relations among these quantities by minimizing the energy at fixed Q.

This is a brutal set of approximations; an infinite-dimensional space of field variations has been chopped down to a mere three-dimensional one. Nevertheless, we will obtain some useful information this way, and we will get back to the full infinite-dimensional space before we're done.

I will also, in the first instance, neglect the contributions to both E and Q coming from the transition zone at the surface of the sphere. This is a mild approximation, at least for large Q, where it is reasonable to expect surface effects to be small corrections to volume ones.

Now for the computation. The exact expression for the energy is

$$E=\int d^3x[\tfrac{1}{2}\dot{\phi}_1^2+\tfrac{1}{2}\dot{\phi}_2^2+\tfrac{1}{2}(\nabla\phi_1)^2+\tfrac{1}{2}(\nabla\phi_2)^2+U]\,. \tag{2.1}$$

In our approximation, this becomes

$$E=\tfrac{1}{2}\omega^2\phi^2V+UV. \tag{2.2}$$

Likewise, in the same approximation, from eqs. (1.3) and (1.4),

$$Q=\omega\phi^2V. \tag{2.3}$$

We wish to minimize E with Q fixed. If we use eq. (2.3) to eliminate ω, we find

$$E=\frac{1}{2}\frac{Q^2}{\phi^2V}+UV. \tag{2.4}$$

As a function of V, this has its minimum at

$$V=Q/\sqrt{2\phi^2U}\,. \tag{2.5}$$

Here,

$$E=Q\sqrt{2U/\phi^2}\,. \tag{2.6}$$

The last step is to minimize this as a function of ϕ. This gives the definition of ϕ_0, eq. (1.6). This completes the computation.

Rough as this calculation was, it did not have among its inputs the assumption that the interior value of ϕ was independent of Q, and thus it is significant that this emerged at the end. It is this that leads to the interpretation of our approximate solution as a lump of Q-matter. It immediately implies that E and V are both proportional to Q, and that ω is independent of Q, and, indeed, equal to ω_0, as defined in eq. (1.8).

Nothing in our computation so far says that a Q-ball has to be spherical; all shapes of the same volume are degenerate in energy. This is because we have neglected the contribution to the integral (2.1) from the transition zone connecting the interior of the Q-ball to the vacuum outside. We would expect this to make a positive contribution to the Q-ball energy proportional to its surface area. (Shortly we shall see how to compute the constant of proportionality, the surface tension of Q-matter.) This lifts the degeneracy and selects among all shapes of the same volume the one of minimum area, to wit, the sphere.

Although a full investigation of Q-ball stability must wait for sect. 3, we are in a position here to investigate some simple decay channels. The most obvious way for a Q-ball to decay is by emitting charged mesons. From eq. (2.6), the energy per unit charge, E/Q, is $\sqrt{2U_0/\phi_0^2}$. Thus a Q-ball is stable under meson emission if this number is less than the meson mass, μ. It is, by eq. (1.6).

Another obvious decay mode is quantum tuneling. Q-matter is much like a false vacuum, in that it is a homogeneous state of nonzero ϕ. We know a false vacuum decays by quantum tunneling; quantum fluctuations produce a bubble of true vacuum inside the false vacuum, which then grows classically. Can a bubble of vacuum similarly appear inside a Q-ball? No, it can not. Quantum tunneling conserves both Q and E. Thus a Q-ball can tunnel only to a state with the same Q and the same E, and, by the arguments I have just given, the only such state is a spherical Q-ball, not a Q-ball with a cavity.

(I remind you that all of these arguments are based on approximations that are valid only in the limit of large Q. They say nothing about the stability, or even the existence, of Q-balls with small values of Q.)

With a little more work we can directly demonstrate that the Q-ball is a solution of the equations of motion, for sufficiently large Q. If we insert eq. (1.7) in the equations of motion, we find

$$\frac{d^2\phi}{dr^2} = -\frac{2}{r}\frac{d\phi}{dr} - \omega^2\phi + U'(\phi)\,. \tag{2.7}$$

This is essentially identical to an equation that occurs in the theory of vacuum decay, and can be treated by methods used there [4].

If we interpret ϕ as a particle position and r as a time, eq. (2.7) can be read as the Newtonian equation of motion for a particle of unit mass subject to viscous damping (with a coefficient inversely proportional to the time) and moving in the potential $\frac{1}{2}\omega^2\phi^2 - U$. This potential is sketched in fig. 2. The sketch is drawn for $\omega_0^2 < \omega^2 < \mu^2$; this is the range in which we shall find solutions. The curve is qualitatively different outside this range: if ω^2 is greater than μ^2, the hill at the origin becomes a valley; if ω^2 is less than ω_0^2, the hill on the right is lower than the hill at the origin. We are searching for a solution in which the particle starts out at time zero at some position, $\phi(0)$, at rest, $d\phi/dr = 0$, and comes to rest at infinite time at $\phi = 0$.

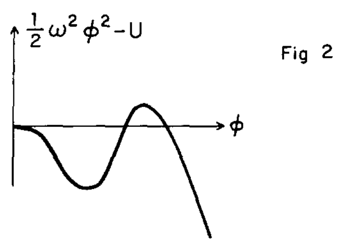

Fig. 2.

I will now argue that such a solution always exists, for ω^2 in the stated range, for an appropriate choice of $\phi(0)$. The argument is based on continuity. I will show that if the particle is released too far from the top of the right-hand hill, it will undershoot, and never reach $\phi = 0$, while if it is released too close to the hilltop, it will overshoot, and reach $\phi = 0$ at finite time with nonzero velocity. Thus, by continuity, there must be some intermediate initial position for which it just manages to reach $\phi = 0$.

To show undershoot is trivial. If the particle is released in a region where the potential is negative, it never has enough energy to climb the hill and reach $\phi = 0$. (The damping force can only diminish the energy.) To show overshoot is hardly more difficult. If we release the particle sufficiently close to the top of the hill, we can arrange that it stays arbitrarily close to the top for arbitrarily long time. At sufficiently long times, the $1/r$ coefficient of the damping force is so small that we can neglect damping; the positive energy of the particle then implies overshoot.

As ω^2 approaches ω_0^2, the top of the right hill approaches ϕ_0, and the potential difference between the two hilltops approaches zero. Thus, in this limit, the particle must spend a very long time close to ϕ_0 before making the transition to $\phi = 0$. This is precisely the description of a large Q-ball obtained from rough approximations at the beginning of this section; we now know that this description corresponds to an exact solution of the equations of motion. (But we do not yet know that this solution is stable.)

We can now get a precise description of the surface of a large Q-ball, and we can compute the surface tension, the surface-area-dependent term in the energy. For a large Q-ball, in the neighborhood of the surface, we can neglect the damping term in the equation of motion; also, we can approximate ω^2 by ω_0^2. Thus, near the surface we must solve

$$\frac{d^2\phi}{dr^2} = \frac{d}{d\phi}[-\tfrac{1}{2}\omega_0^2\phi^2 + U] \equiv \frac{d}{d\phi}\,\hat{U}(\phi)\,. \tag{2.8}$$

A first integral of this is

$$\frac{1}{2}\left(\frac{d\phi}{dr}\right)^2 - \hat{U} = 0\,. \tag{2.9}$$

(The integral must vanish because ϕ goes to zero as r goes to infinity.) Thus

$$R - r = \int_{\bar{\phi}}^{\phi} \sqrt{2\hat{U}}\, d\phi\,. \tag{2.10}$$

Here R is the radius of the Q-ball, the place where $\phi = \bar{\phi}$. Of course, we are free to define the radius to be anywhere we want inside the somewhat fuzzy surface, free to choose $\bar{\phi}$ to be anywhere between 0 and ϕ_0.

For our purposes, it will be convenient to define $\bar{\phi}$ by demanding that

$$\int d^3x\,\phi^2 = \tfrac{4}{3}\pi R^3 \phi_0^2\,. \tag{2.11}$$

Let me show that this indeed defines a choice of $\bar{\phi}$ independent of R, for large R. Eq. (2.11) can be written

$$\int d^3x\,[\phi^2 - \phi_0^2\theta(R-r)] = 0\,. \tag{2.12}$$

If $f(r)$ is some function that is concentrated at r near R, then for large R we can make the approximation

$$\int d^3x f = 4\varepsilon R^2 \int_{-\infty}^{\infty} f(r)\,dr\,. \tag{2.13}$$

In this approximation, eq. (2.12) becomes

$$\int_{-\infty}^{\infty} dr\,[\phi^2 - \phi_0^2\theta(R-r)] = 0\,. \tag{2.14}$$

If we shift the integration variable from r to $r-R$, we see that this condition is independent of R.

We can now compute Q and E as functions of R. By eq. (2.11),

$$Q = \tfrac{4}{3}\pi R^3 \phi_0^2 \omega_0\,. \tag{2.15}$$

I will write E as $E_{\text{surface}} + E_{\text{volume}}$, where

$$E_{\text{surface}} = \int d^3x \left[\frac{1}{2}\left(\frac{d\phi}{dr}\right)^2 + \hat{U}\right], \tag{2.16}$$

$$E_{\text{volume}} = \int d^3x\,\omega_0^2\phi^2\,. \tag{2.17}$$

Again, by eq. (2.11),

$$E_{\text{volume}} = \tfrac{4}{3}\pi R^3 \phi_0^2 = \tfrac{8}{3}\pi R^3 U_0\,. \tag{2.18}$$

For the surface energy, we can use the approximation (2.13),

$$\begin{aligned} E_{\text{surface}} &= 4\pi R^2 \int dr \left[\frac{1}{2}\left(\frac{d\phi}{dr}\right)^2 + \hat{U}\right] \\ &= 4\pi R^2 \left[\int_0^{\phi_0} d\phi\,\sqrt{2\hat{U}}\right], \end{aligned} \tag{2.19}$$

where eq. (2.9) has been used at the last step. The quantity in square brackets is the surface-tension coefficent – what we were after.

3. Stability and existence theorems

In this section I will prove two theorems that will establish the existence of Q-balls and their stability, for sufficiently large Q.

The first theorem has to do with initial-value data, the fields and their time derivatives at fixed time. I will define a set of initial-value data to be of Q-ball type if $\phi_1 = \phi(r)$, $\phi_2 = 0$, $\dot\phi_1 = 0$, and $\dot\phi_2 = \omega\phi(r)$, where ω is a constant and $\phi(r)$ is a positive function monotone decreasing to zero as r goes to infinity.

Theorem 1: For any theory of the type (1.1), with $U \geq 0$, given some set of initial-value data, with some Q and E, there is a set of initial-value data of Q-ball type with the same Q and lesser or equal E.

This theorem will be an important first step in our investigation, but it is not sufficient to prove the existence of Q-balls. For example, it applies to a free field theory, for which there are certainly no Q-balls. We need a stronger theorem that holds for a more restricted class of interactions.

Definition: I will say an interaction, $U(\phi)$, is "acceptable" if

(i) $U(0) = 0$ and U is positive everywhere else. U is twice continuously differentiable. $U'(0) = 0$. $U''(0) = \mu^2$ is positive. (These are just the standard conditions for the existence of unbroken symmetry and massive mesons.)

(ii) The minimum of U/ϕ^2 is attained at some $\phi_0 \neq 0$. (This is just our old condition for Q-balls, eq. (1.6).)

(iii) There exist three positive numbers, a, b, and c, with $c > 2$, such that

$$\tfrac{1}{2}\mu^2\phi^2 - U \leq \min(a, b\phi^c)\,. \tag{3.1}$$

(Condition (ii) implies that U must somewhere dip below $\frac{1}{2}\mu^2\phi^2$. Condition (iii) is a mild technical restriction saying it should not dip too far, either at infinity or at zero.)

Theorem 2: If U is acceptable, there exists $Q_{\min} \leq 0$, such that for any $Q > Q_{\min}$, there is initial-value data of Q-ball type that minimizes E for that value of Q. Furthermore, this is the initial-value data for a Q-ball solution of the equations of motion, that is to say, it obeys eq. (2.7).

This theorem guarantees both the existence and the absolute stability of Q-balls. You might think that there is no need for theorem 2, since the arguments of sect. 2 show the existence of solutions of eq. (2.7). However, although these are easily shown to be stationary points of E at fixed Q, there is no guarantee that they are minima. Indeed, in the absence of theorem 2, we have no guarantee that minima exist. For example, for free field theory, $U = \frac{1}{2}\mu^2\phi^2$, the infimum of E for fixed Q is $\mu|Q|$. Although we can come arbitrarily close to this lower bound, there is no set of initial-value data that attains it. (Proof: To attain the bound, ϕ would have to have vanishing gradient, which precludes finite nonzero Q.)

3.1. PROOF OF THEOREM 1

Let us assemble ϕ_1 and ϕ_2 into a two-vector, $\boldsymbol{\phi}$. The energy can be written as

$$E = \int \mathrm{d}^3x\,[\tfrac{1}{2}\dot{\boldsymbol{\phi}}\cdot\dot{\boldsymbol{\phi}} + \tfrac{1}{2}\nabla\boldsymbol{\phi}\cdot\nabla\boldsymbol{\phi} + U]\,, \tag{3.2}$$

while

$$Q = i \int d^3x\, \dot{\boldsymbol{\phi}} \cdot \sigma_y \boldsymbol{\phi}\,, \tag{3.3}$$

where σ_y is the usual Pauli matrix. We define

$$I \equiv \int d^3x\, \boldsymbol{\phi} \cdot \boldsymbol{\phi} = \int d^3x\, \sigma_y \boldsymbol{\phi} \cdot \sigma_y \boldsymbol{\phi}\,, \tag{3.4}$$

and

$$\omega \equiv Q/I\,. \tag{3.5}$$

(I is the internal-space analog of the moment of inertia, whence its name.) By the Schwarz inequality,

$$Q^2 \leqslant I \int d^3x\, \dot{\boldsymbol{\phi}} \cdot \dot{\boldsymbol{\phi}}\,. \tag{3.6}$$

This gives a lower bound on the $\dot{\boldsymbol{\phi}}$ term in E for fixed $\boldsymbol{\phi}$ and Q. Furthermore, again by the Schwarz inequality, this bound is saturated if and only if

$$\dot{\boldsymbol{\phi}} = -i\omega\sigma_y \boldsymbol{\phi}\,. \tag{3.7}$$

In the sequel, I will assume $\dot{\boldsymbol{\phi}}$ has been chosen to obey this equation. E can then we written as a functional of $\boldsymbol{\phi}$ only, with Q as a parameter,

$$E_Q = \int d^3x\, [\tfrac{1}{2}\nabla\boldsymbol{\phi} \cdot \nabla\boldsymbol{\phi} + U] + \frac{Q^2}{2I}\,. \tag{3.8}$$

Let us define angular variables in internal space, ϕ and θ, by

$$\phi_1 = \phi \cos\theta\,, \qquad \phi_2 = \phi \sin\theta\,, \tag{3.9}$$

with $\phi \geqslant 0$. Only one term in eq. (3.8) depends on θ, the derivative term,

$$E_Q = \int d^3x\, \tfrac{1}{2}\phi^2 (\nabla\theta)^2 + \cdots\,. \tag{3.10}$$

Thus we can always minimize the energy by keeping ϕ fixed and making θ a constant. With no loss of generality, we can take this constant to be zero. Thus,

$$\phi_1 = \phi\,, \qquad \phi_2 = 0\,, \tag{3.11}$$

and, by eq. (3.7),

$$\dot{\phi}_1 = 0\,, \qquad \dot{\phi}_2 = \omega\phi\,. \tag{3.12}$$

For any positive function of position, $\phi(x)$, vanishing at infinity, the spherical rearrangement of the function, $\phi_R(r)$, is defined as the spherically symmetric monotone decreasing function obeying

$$\mu_L\{x|\phi_R(x) \geqslant \varepsilon\} = \mu_L\{x|\phi) \geqslant \varepsilon\} \text{ for any } \varepsilon > 0\,, \tag{3.13}$$

where μ_L denotes the Lebesque measure. It is immediate that I and the integral of U are unchanged if we replace ϕ by ϕ_R. The spherical rearrangement theorem of Glaser, Grosse, Martin and Thirring [5] states that

$$\int d^3x\,(\nabla\phi)^2 \geqslant \int d^3x\,(\nabla\phi_R)^2\,, \tag{3.14}$$

with equality attained only if $\phi = \phi_R$. Thus we can always minimize the energy by choosing ϕ to be spherically symmetric and monotone decreasing. This, together with eqs. (3.11) and (3.12), completes the proof of theorem 1.

3.2. PROOF OF THEOREM 2

I will now show that for any positive Q such that

$$\inf E_Q < \mu Q\,, \tag{3.15}$$

there exist Q-balls. We know that for any acceptable interaction this condition is satisfied for sufficiently large Q, because we can use the approximate solutions of sect. 2 as initial-value data. We also know that if it is satisfied for any Q, it is satisfied for any Q' greater than Q, because we can always add mesons at rest at infinity. By this I mean we can change the fields in the following way:

$$\phi_1(\boldsymbol{x}) \to \phi_1(\boldsymbol{x}) + L^{-3/2} f\left(\frac{\boldsymbol{x}-\boldsymbol{d}}{L}\right), \tag{3.16}$$

with f an arbitrary function of compact support, and with like changes for the other pieces of initial-value data. As L and $\boldsymbol{d}$ grow arbitrarily large, only the quadratic terms free of spatial derivatives in the energy survive. Thus we can arrange to add $Q'-Q$ to Q while adding $\mu(Q'-Q)$ to E. Thus, there must exist a $Q_{\min}$ such that eq. (3.15) holds for any Q greater than $Q_{\min}$, and for no Q less than $Q_{\min}$.

In the body of the proof, I'll have to write E_Q as a sum of terms in several different ways. I'll define all the needed decompositions here. The first decomposition is

$$E_Q = K + V + Q^2/2I\,, \tag{3.17}$$

where

$$K = \tfrac{1}{2}\int d^3x\,(\nabla\phi)^2\,, \tag{3.18}$$

and

$$V = \int d^3x\, U(\phi)\,. \tag{3.19}$$

(Please do not confuse this with V, the Q-ball volume, in sect. 2.) Another decomposition is

$$E_Q = K + W + \tfrac{1}{2}\mu^2 I + Q^2/2I\,, \tag{3.20}$$

where

$$W = \int d^3\boldsymbol{x}\,[U - \tfrac{1}{2}\mu^2\phi^2]\,. \tag{3.21}$$

Finally, we'll need the decomposition of the integrand in this integral into its positive and negative parts

$$U - \tfrac{1}{2}\mu^2\phi^2 \equiv w = w_+ + w_-\,, \tag{3.22}$$

where, as usual, $w_\pm = w\theta(\pm w)$. Eq. (3.1) can be rephrased as

$$|w_-| < \min(a, b\phi^c)\,. \tag{3.23}$$

I'll denote the integrals of these quantities by $W_\pm$.

I begin the proof proper by constructing a minimizing sequence, an infinite sequence of functions, ϕ_n, such that $\lim E_Q[\phi_n] = \inf E_Q$. The existence of such a sequence is guaranteed by the definition of the infimum. By theorem 1, we can choose the sequence to be of Q-ball type, that is to say, $\phi_1 = \phi(r)$, $\phi_2 = 0$, with $\phi(r)$ monotone decreasing to zero. Furthermore, we can choose the functions in the sequence is guaranteed by the definition of the infimum. By theorem 1, we can without bound as n goes to infinity).

K is a positive quantity bounded above by E_Q and therefore there exists a subsequence such that K has a limit. An identical argument applies to V. If K, V, and E converge, so does I, by subtraction, and so does W, by further subtraction. I will denote the limiting values of these quantities by overbars. Thus

$$\lim_{n\to\infty} K[\phi_n] = \bar{K}\,, \tag{3.24}$$

with similar equations for $\bar{V}$, $\bar{I}$, $\bar{W}$, and $\bar{E} = \inf E_Q$. Because these limits exist, we can always choose a subsequence such that K, V, I, W, and E are bounded uniformly in n. I will assume this has been done.

Later on we will need a lower bound on $\bar{I}$.

$$\bar{I} > Q/\mu\,. \tag{3.25}$$

Proof: From eq. (3.20),

$$\bar{E} - \bar{K} - \bar{W} = \tfrac{1}{2}(\mu^2\bar{I} + Q^2/\bar{I})\,. \tag{3.26}$$

The right-hand side of this equation has its minimum at $\bar{I} = Q/\mu$, where it equals μQ. Now, we can always increase $\bar{I}$, without affecting $\bar{W}$ and $\bar{K}$, by adding mesons at rest at infinity, as in eq. (3.16), to the elements of the minimizing sequence. Thus, if $\bar{I}$ is strictly less than Q/μ, we can in this way construct a sequence that converges to a lower energy than the infimum – an absurdity. It remains to elminate $\bar{I} = Q/\mu$. In this case, from eqs. (3.26) and (3.15),

$$\bar{W} < -\bar{K}\,. \tag{3.27}$$

If we consider a scale transformation,

$$\phi_n(\boldsymbol{x}) \to \phi_n(\boldsymbol{x}(1+\alpha)), \tag{3.28}$$

with α a small parameter, we find

$$\bar{E} \to \bar{E} - \alpha\bar{K} - 3\alpha\bar{W} + \mathrm{O}(\alpha^2). \tag{3.29}$$

(There is no $\bar{I}$ term because $\bar{I}$ is as at a stationary point.) Thus, by choosing α small and negative, we can again absurdly lower the energy below the infimum. This completes the proof of eq. (3.25).

Let us define f_n to be $r\phi_n$. Then

$$K[\phi_n] = 4\pi \int_0^\infty \mathrm{d}r \left(\frac{\mathrm{d}f_n}{\mathrm{d}r}\right)^2, \tag{3.30}$$

$$I[\phi_n] = 4\pi \int_0^\infty \mathrm{d}r f_n^2. \tag{3.31}$$

Since f_n vanishes at infinity,

$$f_n^2(r) = -\tfrac{1}{2}\int_r^\infty \mathrm{d}r f_n \frac{\mathrm{d}f_n}{\mathrm{d}r} \leqslant \frac{1}{8\pi}(K[\phi_n]I[\phi_n])^{1/2} \tag{3.32}$$

By the Schwarz inequality. Likewise,

$$|f_n(r_1) - f_n(r_2)| = \left|\int_{r_1}^{r_2} \mathrm{d}r \frac{\mathrm{d}f_n}{\mathrm{d}r}\right| \leqslant |r_1 - r_2|^{1/2}(K[\phi_n]/4\pi)^{1/2}. \tag{3.33}$$

Thus the f's form a uniformly bounded sequence of equicontinuous functions. Therefore, by Ascoli's theorem, they contain a subsequence which is pointwise convergent everywhere and uniformly convergent on any finite interval. This implies the same is true of the ϕ's, except possibly at $r=0$. I will denote the limit of this convergent subsequence as Ψ. (The natural choice would be $\bar{\phi}$, but this has been used for another purpose in sect. 2.) The task is to show that this is the sought-after Q-ball, that $E_Q[\Psi]$ is $\bar{E}$.

K defines a Hilbert-space norm under which the ϕ's are a bounded family of vectors. Such a bounded family always has a weakly-convergent subsequence. The norm of the weak limit is always less than or equal to the limit of the norms. Thus,

$$K[\Psi] \leqslant \bar{K}. \tag{3.34}$$

By the same reasoning,

$$I[\Psi] \leqslant \bar{I}. \tag{3.35}$$

W requires more work, because it is the sum of positive and negative parts, which must be treated separately. For any positive r_-,

$$\left|\int_{r \leqslant r_-} \mathrm{d}^3x\, w_-(\phi_n)\right| \leqslant \tfrac{4}{3}\pi r_-^3 a, \tag{3.36}$$

and

$$\left|\int_{r\geq r_+} d^3x\, w_-(\phi_n)\right| \leq b\left|\frac{\sup f_n}{r_+}\right|^{c-2} I[\phi_n]\,. \tag{3.37}$$

Thus, by proper choice of r_- and r_+, we can arrange that the contribution to W_- from r outside the interval $[r_-, r_+]$ is arbitrarily small, uniformly in n. Since ϕ_n converges uniformly to Ψ in this interval,

$$\lim_{n\to\infty} W_-[\phi_n] = W_-[\Psi]\,. \tag{3.38}$$

W_+ is the integral of a positive integrand. Hence, by Fatou's lemma,

$$W_+[\Psi] \leq \liminf_{n\to\infty} W_+[\phi_n]\,. \tag{3.39}$$

Thus,

$$W[\Psi] \leq \tilde{W}\,. \tag{3.40}$$

I will now improve eq. (3.35), replacing the inequality by an equality. Let us assume that $I[\Psi]$ is strictly less than $\bar{I}$. Then, by adding mesons at rest at infinity, as in eq. (3.16), we can construct a new function, ϕ', such that $K[\phi'] = K[\Psi]$, $W[\phi'] = W[\Psi]$ and $I[\phi']$ is anywhere we please between $I[\Psi]$ and $\bar{I}$. In particular, by eq. (3.25), we can arrange matters such that

$$\bar{I} > I[\phi'] > Q/\mu\,. \tag{3.41}$$

This implies

$$\tfrac{1}{2}(\mu^2 I[\phi'] + Q^2/I[\phi']) < \tfrac{1}{2}(\mu^2\bar{I} + Q^2/\bar{I})\,. \tag{3.42}$$

Combining this with eqs. (3.37) and (3.40), we find that $E_Q[\phi']$ is strictly less than $\bar{E}$. Since $\bar{E}$ is inf E_Q, this is impossible.

Thus, $I[\Psi] = \bar{I}$. Again, from eqs. (3.37) and (3.40), this implies that $E_Q[\Psi] \leq \bar{E}$. Less is impossible; thus,

$$E_Q[\Psi] = \inf E_Q\,. \tag{3.43}$$

This proves the first part of theorem 2. However, the proof of the second part is now immediate: since Ψ is the absolute minimum of E_Q, *a fortiori* it is stationary point of E_Q. But

$$\frac{\delta E_Q}{\delta\Psi} = -\nabla^2\Psi + U'(\Psi) - Q^2\Psi/\bar{I}^2\,. \tag{3.44}$$

The vanishing of this expression *is* the equation of motion for steady rotation in internal space, if we identify $Q/\bar{I}$ as ω.

This completes the proof of theorem 2.

4. Exciting Q-balls

In this section I find those small vibrations of Q-balls whose frequencies go to zero as R goes to infinity. Upon quantization, these become the excitation levels of lowest energy for large R. In the limit of infinite R, any such family of vibrations must come down to a vibration of zero frequency. By identifying these zero modes we know where to look for the desired vibrations.

If we consider infinite space filled with Q-matter, there is an obvious zero mode generated by infinitesimal Q rotations. As we shall see, this is the zero wave-vector limit of a sound wave, which, for small wave-vector, $\boldsymbol{k}$, has frequency proportional to $|\boldsymbol{k}|$. For a finite Q ball, $|\boldsymbol{k}|$ takes discrete values proportional to $1/R$. Thus we have a spectrum of excitations (phonons) with energies proportional to $1/R$.

Another way of going to the infinite-R limit is to sit, not at the center of the Q-ball, but at its surface. The limit is then a half-space filled with Q-matter, separated from the vacuum by a planar interface, the limit of the Q-ball surface. Again, there is an obvious zero mode, associated with translations of the interface normal to itself. As we shall see, this is the zero wave-vector limit of a surface wave, which, for small $|\boldsymbol{k}|$, has frequency proportional to $|\boldsymbol{k}|^{3/2}$. Thus we have a spectrum of excitations with energies proportional to $1/R^{3/2}$. In the case of interest, R is much larger than the natural length scales of the theory (e.g. meson Compton wavelengths). Thus, the surface excitations have much lower energies than the phonons.

I'll first find the spectrum of the sound waves, by studying small perturbtions about Q-matter, the homogeneous solution to the field equations. Then I'll rederive the same spectrum by the methods of classical fluid dynamics, exploiting the local conservtion of energy, momentum, and Q. Finally, I'll go on to find the surface-wave spectrum by fluid-dynamics methods; this turns out to be a simpler way to attack the problem than a direct assault on the equations of motion.

It is convenient to study small perturbations about Q-matter in internally corotating coordinates. I define δ_1 and δ_2 by

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}=\begin{pmatrix}\cos\omega_0 t & -\sin\omega_0 t\\ \sin\omega_0 t & \cos\omega_0 t\end{pmatrix}\begin{pmatrix}\phi_0+\delta_1\\ \delta_2\end{pmatrix}. \tag{4.1}$$

If we insert this into the equations of motion, and work only to first order in the δ's, we find

$$\ddot{\delta}_1+2\omega_0\dot{\delta}_2-\nabla^2\delta_1+U_0''\delta_1-\omega_0^2\delta_1=0\,, \tag{4.2a}$$

$$\ddot{\delta}_2-2\omega_0\dot{\delta}_1-\nabla^2\delta_2+U_0'\delta_2/\phi_0-\omega_0^2\delta_2=0\,, \tag{4.2b}$$

where a zero subscript denotes a quantity evaluated at $\phi=\phi_0$. Because ϕ_0 is a stationary point of U/ϕ^2,

$$U_0'/\phi_0=2U_0/\phi_0^2=\omega_0^2 \tag{4.3}$$

and we can drop the last two terms in eq. (4.2b).

The eqs. (4.2) are invariant under space-time translations. Thus the normal modes can be chosen to be proportional to $\exp(ik\cdot x)$, where k is a zero of the determinant

$$\begin{vmatrix} -(k^0)^2+|\boldsymbol{k}|^2+U_0''-\omega_0^2 & 2i\omega_0 k^0 \\ -2i\omega_0 k^0 & -(k^0)^2+|\boldsymbol{k}|^2 \end{vmatrix} . \tag{4.4}$$

Retaining only the leading terms for small k^0 and $\boldsymbol{k}$, we find

$$(U_0''-\omega_0^2)|\boldsymbol{k}|^2=(U_0''+3\omega_0^2)(k^0)^2 . \tag{4.5}$$

As promised, this is an acoustic dispersion equation, with the velocity of sound given by

$$v_S^2=\frac{U_0''-\omega_0^2}{U_0''+3\omega_0^2} . \tag{4.6}$$

(Because ϕ_0 minimizes U/ϕ^2, $(U_0''-\omega_0^2)=\phi_0^2(U/\phi_0^2)''$ is always positive.)

I will now obtain the same result by the methods of classical fluid dynamics. I'll begin by reviewing the standard derivation of the velocity of sound for a relativistic perfect fluid [6].

Fluid dynamics is based on the conservation of the particle-number current,

$$\partial_\mu j^\mu=\partial_\mu(nu^\mu)=0 \tag{4.7}$$

and the conservation of the energy-momentum tensor,

$$\partial_\mu T^{\mu\nu}=\partial_\mu[(e+p)u^\mu u^\nu-pg^{\mu\nu}]=0 . \tag{4.8}$$

Here e, p, and n, are the rest-frame energy density, pressure, and particle-number density, and u is the fluid four-velocity, a unit timelike vector. In addition, we need the equation of state of the fluid, e as a function of n. From this, the dependence of p on n can be deduced from the first law of thermodynamics.

(Actually, in traditional fluid mechanics, e and p depend not just on n, but on the rest-frame entropy density, s. What I have defined here is a cold fluid, $s=0$ everywhere. Q-matter turns out to be this special kind of fluid, with particle number replaced by Q, as we shall see shortly.)

Let us consider a small deviation from a homogeneous equilibrium state of the fluid at rest. Thus, $e=e_0+\delta e$. where e_0 is the equilibrium value and δe is the deviation, to be treated in first order; δp and δn are defined in a like way. Also, u^i is of first order, and $u^0=1$, neglecting terms of higher order.

If we retain only terms of first order, the conservation equations, (4.7) and (4.8), become

$$\delta\dot{n}+n_0\partial_i u^i=0 , \tag{4.9a}$$

$$\delta\dot{e}+(e_0+p_0)\partial_i u^i=0 , \tag{4.9b}$$

$$(e_0+p_0)\dot{u}^i-\partial^i\delta p=0 . \tag{4.9c}$$

If we eliminate u^i from the last two equations, we find

$$\delta\ddot{e} = \nabla^2\delta p = \left(\frac{\mathrm{d}p}{\mathrm{d}e}\right)_0 \nabla^2\delta e\,. \tag{4.10}$$

This is the wave equation, with the velocity of sound given by

$$v_\mathrm{S}^2 = \left(\frac{\mathrm{d}p}{\mathrm{d}e}\right)_0\,. \tag{4.11}$$

This completes the review.

To apply this formalism to Q-matter, we need to find the Q-matter equation of state, e as a function of n (now Q-density), in a spatially homogeneous state where the fluid is at rest. In this context, "at rest" means that the only nonzero component of j^μ is the time component; that is to say, ϕ is a constant, and the fields are in steady rotation in internal space with some frequency ω.

For such a state, the field equations become

$$\omega^2\phi = U'(\phi)\,. \tag{4.12}$$

This fixes ω in terms of ϕ. (Provided U' is positive.) Thus,

$$n = \omega\phi^2 = \sqrt{U'\phi^3}\,. \tag{4.13}$$

As advertised, the states are labeled by n alone, like those of a cold fluid. By comparing the energy-momentum tensor of a field theory,

$$T^{\mu\nu} = \partial^\mu\boldsymbol{\phi}\cdot\partial^\nu\boldsymbol{\phi} - g^{\mu\nu}\mathscr{L}\,, \tag{4.14}$$

to that of a fluid at rest, eq. (4.8), we find that

$$e = \tfrac{1}{2}\omega^2\phi^2 + U = \tfrac{1}{2}U'\phi + U\,, \tag{4.15}$$

$$p = \tfrac{1}{2}\omega^2\phi^2 - U = \tfrac{1}{2}U'\phi - U\,. \tag{4.16}$$

Note that when $\phi = \phi_0$, $p = 0$; the ground state of Q-matter wants neither to expand nor contract.

We can now compute the velocity of sound in the ground state.

$$v_\mathrm{S}^2 = \left(\frac{\mathrm{d}p}{\mathrm{d}e}\right)_0 = \frac{U_0''\phi_0 - U_0'}{U_0''\phi_0 + 3U_0'}\,. \tag{4.17}$$

By eq. (4.3), this is the same as the result of the previous method, eq. (4.6).

To extend these methods to surface waves*, we need the contribution of the Q-ball surface to the energy-momentum tensor. Because we are studying waves of very long wavelength, it is reasonable to approximate the surface as being infinitely

* I doubt that the following theory of relativistic capillary waves is original, but I have not been able to find it in the literature. In any case, the dispersion equations it finally reaches, eqs. (4.40) and (4.41), are obvious generalizations of the standard nonrelativistic equations [7].

thin. Thus, for example, for a surface occupying the plane $x^3 = 0$, the energy density is approximated as

$$T^{00}_{\text{surface}} = \alpha\delta(x^3)\,, \tag{4.18}$$

where α is the surface-tension coefficient computed at the end of sect. 2,

$$\alpha = \int_0^{\phi_0} \mathrm{d}\phi[2U - \omega_0^2\phi^2]^{1/2}\,. \tag{4.19}$$

It is straightforward to use eq. (4.14) to compute the other components of $T^{\mu\nu}_{\text{surface}}$, in the same approximation. The only nonzero components are

$$T^{11}_{\text{surface}} = T^{22}_{\text{surface}} = -T^{00}_{\text{surface}}\,. \tag{4.20}$$

This can be written as

$$T^{\mu\nu}_{\text{surface}} = \alpha[g^{\mu\nu} + n^\mu n^\nu]\delta_{\text{S}}(x)\,. \tag{4.21}$$

Here n^μ is the unit space-like normal vector to the surface, and $\delta_{\text{S}}(x)$ is the delta-function concentrated on the surface. In this form, this is the right expression for a general surface, not just a hyperplane. (For a general surface, δ_{S} is normalized such that its integral over any volume is the surface area of the portion of the surface contained in the volume.) This is because we are studying processes that take place on very large space and time scales compared to the scales characteristic of the internal dynamics of the surface. Thus, at every point, the surface looks as if it were part of a planar interface in equilibrium, as rest in some Lorentz frame; it's just that the rest frame changes from point to point.

It will be convenient to introduce the characteristic function of the Q-ball, $\chi(x)$, defined by $\chi = 1$ inside the Q-ball and $\chi = 0$ outside. χ is related to n^μ and δ_{S} by

$$\partial_\mu\chi = n_\mu\delta_{\text{S}}\,. \tag{4.22}$$

In terms of these,

$$T^{\mu\nu} = [(e+p)u^\mu u^\nu - g^{\mu\nu}p]\chi + \alpha[g^{\mu\nu} + n^\mu n^\nu]\delta_{\text{S}}\,. \tag{4.23}$$

Also,

$$j^\mu = nu^\mu\chi\,. \tag{4.24}$$

(In the limit of zero thickness, the surface contains no Q.)

Thus, the terms that appear on the right-hand side of the conservation laws fall into two sets. First, there are terms that multiply χ. The vanishing of these terms simply recreates the usual hydrodynamic equation inside the fluid; they make no reference to the surface. Second, there are terms that multiply δ_{S}. The vanishing of these terms gives us new information at the surface.

From eq. (4.24), we find

$$u^\mu n_\mu\delta_{\text{S}} = 0\,. \tag{4.25}$$

From eq. (4.23), we find

$$(-pn^\nu + \alpha\partial_\mu n^\mu n^\nu + \alpha n^\mu \partial_\mu n^\nu)\delta_S = 0\,. \tag{4.26}$$

If we dot n_ν into this, and use $n_\nu n^\nu = -1$, we find

$$(-p + \alpha\partial_\mu n^\mu)\delta_S = 0\,. \tag{4.27}$$

Eqs. (4.25) and (4.27) are the fundamental equations for the motion of the surface.

We can now analyze surface waves. Let us consider a Q-ball (of infinite radius) which occupies all points obeying

$$x^3 \leq \eta(x^0, x^1, x^2)\,. \tag{4.28}$$

I will work to first order in η. To this order,

$$\chi = \theta(\eta - x^3)\,, \tag{4.29a}$$

$$n_\mu = \partial_\mu(\eta - x^3)\,. \tag{4.29b}$$

The fundamental surface equations become

$$\dot\eta = u^3 \qquad \text{at } x^3 = 0\,, \tag{4.30}$$

$$\alpha\partial_\mu\partial^\mu\eta = \delta p \qquad \text{at } x^3 = 0\,, \tag{4.31}$$

Now to find solutions. By translational invariances of the problem, we can always choose our normal modes to be of the form

$$\delta p = f(x^3)\exp(ik^0x^0 - i\boldsymbol{k}_\perp \cdot \boldsymbol{x}_\perp)\,, \tag{4.32}$$

where $\boldsymbol{x}_\perp$ denotes the part of $\boldsymbol{x}$ lying in the 1-2 plane. From the wave equation, eq. (4.10),

$$[v_S^2\partial_3^2 + (k^0)^2 - v_S^2|k_\perp|^2]f = 0\,. \tag{4.33}$$

Thus if $v_S^2|\boldsymbol{k}_\perp|^2$ is less than $(k^0)^2$ we have oscillatory behavior in x^3. These are sound waves (reflected at the surface), of no interest to us at the moment. But if $v_S^2|\boldsymbol{k}_\perp|^2$ is greater than $(k^0)^2$, we can have damped behavior

$$f \propto e^{\kappa x^3}\,, \tag{4.34}$$

with

$$v_S^2\kappa^2 = v_S^2|\boldsymbol{k}_\perp|^2 - (k^0)^2. \tag{4.35}$$

These are the surface waves.

Eqs. (4.9c) and (4.30) enable us to connect η with δp (at $x^3 = 0$),

$$(e_0 + p_0)(k^0)^2\eta - \kappa\delta p = 0\,. \tag{4.36}$$

The same two quantities are connected by eq. (4.31),

$$\delta p = -\alpha[(k^0)^2 - |\boldsymbol{k}_\perp|^2]\eta\,. \tag{4.37}$$

Thus

$$e_0(k^0)^2+\alpha\kappa[(k^0)^2-|\boldsymbol{k}_\perp|^2]=0\,. \tag{4.38}$$

(Here I have used that $p_0=0$.) Eliminating κ with eq. (4.35), we find

$$v_S^2e_0^2(k^0)^4=\alpha^2[v_S^2|\boldsymbol{k}_\perp|^2-(k^0)^2][(k^0)^2-|\boldsymbol{k}_\perp|^2]^2\,. \tag{4.39}$$

We wish to find k^0 for small $|\boldsymbol{k}_\perp|$. If k^0 goes to zero linearly with $|\boldsymbol{k}_\perp|$ (or less rapidly), the right-hand side of this equation is $0(k^0)^6$, while the left-hand side is $0(k^0)^4$. Thus, k^0 must go to zero more rapidly than $|\boldsymbol{k}_\perp|$, and for small $|\boldsymbol{k}_\perp|$ we may neglect k^0 on the right. Thus,

$$(k^0)^2=\frac{\alpha}{e_0}|\boldsymbol{k}_\perp|^3\,. \tag{4.40}$$

This is the promised dispersion equation.

It is possible to obtain a somewhat more refined estimate of the surface-wave frequencies by applying the hydrodynamic equations to a Q-ball of finite radius. The computation is essentially a rerun of the one we have just done, with $R+\eta-r$ replacing $\eta-x^3$. I will merely report the result here. The surface waves are of the form $\eta=Y_{lm}(\theta,\phi)$, with $l=1,2,\ldots,$ and

$$(k^0)^2=\frac{\alpha}{e_0R^3}l(l+2)(l-1)\,. \tag{4.41}$$

The zero mode with $l=1$ is, of course, just translation of the Q-ball.

I have profited from discussions with Luis Alvarez-Gaumé, Minos Axenides, Eddie Farhi, Paul Ginsparg, Sheldon Glashow, Greg Moore, Joe Polchinski, and Alex Safian. Much of this work was done while I was resident at the Aspen Center for Physics; I thank the Center for its hospitality.

Appendix added in proof

When I wrote this paper, I was ignorant of several relevant earlier papers. My subsequent enlightenment is due to correspondence from J.D. Breit and P. Mathieu. Thanks to their efforts, I now know that:

(1) In the course of his work on nontopological solitons, T.D. Lee constructed solutions to a family of theories of a single complex scalar field in one space dimension [8]. The Q-balls I have constructed reduce to these solutions when the number of space dimensions is reduced to one.

(2) There is a large literature on the construction of Q-ball-like solutions (in three dimensions) for theories for which $U'(0)$ is infinite. (For example, U could be proportional to ϕ^a for small ϕ, with a between zero and one.) The singularity of U' has the effect of pushing the meson mass up to infinity; this greatly simplifies

consideration of stability. The key papers here seem to me to be those of Werle and Morris [9].

(3) Almost two decades ago, Rosen showed the existence of Q-ball solutions for a class of U's included in the class considered here [10]. So far as I know, he did not address the question of stability.

References

[1] T.D. Lee and G.C. Wick, Phys. Rev. D9 (1974) 2291;
R. Friedberg, T.D. Lee, and A. Sirlin, Phys. Rev. D13 (1976) 2739, Nucl. Phys. B115 (1976) 1, 32;
R. Friedberg and T.D. Lee, Phys. Rev. D15 (1977) 1694

[2] M. Axenides and A. Safian, private communication

[3] A. Guth and E. Weinberg, Nucl. Phys. B212 (1983) 321

[4] S. Coleman, Phys. Rev. D15 (1977) 2929;
S. Coleman, V. Glaser, and A. Martin, Comm. Math. Phys. 58 (1978) 211

[5] V. Glaser, H. Grosse, A. Martin, and W. Thirring, Studies in mathematical physics, eds. E. Lieb, B. Simon, and A. Wightman (Princeton Univ. 1976) 169

[6] L. Landau and E. Lifshitz, Fluid mechanics (Pergamon, London, 1959) 499

[7] Ref. [6], 237

[8] T.D. Lee, in Proc. of the Symposium on frontier problems in high energy physics, ed. L. Foa and L.A. Radicatti (Scuola Normale Superiore, Pisa, 1976) 47;
T.D. Lee, Particle physics and introduction to field theory (Harwood, London, 1981) 141

[9] J. Werle, Phys. Lett. 71B (1977) 367;
T.F. Morris, Phys. Lett. 76B (1978) 337; 78B (1978) 87

[10] G. Rosen, J. Math. Phys. 9 (1968) 996