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Construction of a Crossing-Simmetric, Regge-Behaved Amplitude for Linearly Rising Trajectories.

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Crossing has been the first ingredient used to make Regge theory a predictive concept in high-energy physics. However, a complete and satisfactory way of imposing crossing and crossed-channel unitarity is still lacking. We can look at the recent investigations on the properties of Reggeization at $t=0$ as giving a first encouraging set of results along this line of thinking ([1]). A technically different approach, based on superconvergence, has been also recently investigated ([2]), and the possibility of a self-consistent determination of the physical parameters, through the use of sum rules, has been stressed.

In this note we propose a quite simple expression for the relativistic scattering amplitude, that obeys the requirements of Regge asymptotics and crossing symmetry in the case of linearly rising trajectories. Its explicit form is suggested by the work of ref. ([3]) and contains only a few free parameters (**).

Our expression contains automatically Regge poles in families of parallel trajectories (at all t) with residue in definite ratios. It furthermore satisfies the conditions of superconvergence ([4]) and exhibits in a nice fashion the duality between Regge poles and resonances in the scattering amplitude.

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([1]) For a general review of these problems see L. Bertocchi: *Proc. of the Heidelberg International Conference on Elementary Particles* (Amsterdam, 1967).

([2]) Such an approach was proposed independently by M. Ademollo, H. R. Rubinstein, G. Veneziano and M. A. Virasoro: *Phys. Rev. Lett.*, **19**, 1402 (1967) and *Phys. Lett.*, **27**, B 99 (1968), and by S. Mandelstam: *Phys. Rev.*, **166**, 1539 (1968). Further developments and a number of references to related works can be found in ref. ([3]).

([3]) M. Ademollo, H. R. Rubinstein, G. Veneziano and M. A. Virasoro: Weizmann Institute preprint (1968), submitted to *Phys. Rev.*

(**) We shall mostly work here in the approximation of real, linear trajectories and consequently of narrow resonances. We briefly discuss the effects of a nonzero imaginary part in the trajectory function which, in any case, we demand to have a linearly rising real part.

([4]) For superconvergence we mean both the original sum rules proposed by V. de Alfaro, S. Fubini, G. Furlan and C. Rossetti: *Phys. Lett.*, **21**, 576 (1966), and the more recent generalized superconvergence (finite-energy) sum rules (see ref. ([3]) for detailed references). A unified treatment of all superconvergence sum rules has been given by S. Fubini: *Nuovo Cimento*, **52** A, 224 (1967).

The first example we want to discuss is the scattering $\pi\pi \to \pi\omega$, whose convenient properties have been already stressed in ref. [3]. We introduce the invariant amplitude $A(s, t, u)$ through the definition of the T-matrix

$$T = \varepsilon_{\mu\nu\varrho\sigma} e_\mu P_{1\nu} P_{2\varrho} P_{3\sigma} \cdot A(s, t, u) , \tag{1}$$

where P_i are the pion momenta and e_μ is the ω polarization vector. $A(s, t, u)$ has only dynamical singularities as it is free of kinematical ones. It is also completely symmetric in the three Mandelstam variables.

It was found in ref. [3] that a « good » parametrization of A at high s and fixed t could be written as

$$A(s, t, u) \underset{s\to\infty}{\simeq} \frac{\bar\beta}{\pi} \Gamma\big(1-\alpha(t)\big)\big(-\alpha(s)\big)^{\alpha(t)-1} + (s \leftrightarrow u) \tag{2}$$

with $\bar\beta =$ const. We use the word « good » in the sense that eq. (2), when used as an input, is able to reproduce itself quite consistently through the use of superconvergence sum rules.

What is the amplitude for nonasymptotic values of s? If eq. (2) were exact after some $\bar s$, analyticity in the s-plane (at fixed t) would require it to be valid at all s and eq. (2) is certainly a solution of superconvergence. However, eq. (2) does not satisfy s, t crossing as this demands poles in s such as those induced in t by the $\Gamma\big(1-\alpha(t)\big)$ factor. On the other hand these poles in s could in principle destroy the asymptotic behaviour (2) through the introduction of fixed singularities. The lowest-moment sum rules are just imposing that this is not happening at the nearest negative integers. Furthermore, we expect that the presence of bumps in the low-energy region will produce (through analyticity) a modification of the high-energy form which will not be as smooth as eq. (2), but will rather show oscillations in s.

Consequently, we take out the factor $\big(-\alpha(s)\big)^{\alpha(t)-1}$ and we symmetrize eq. (2) multiplying by a factor $\Gamma\big(1-\alpha(s)\big)$ and dividing by $\Gamma\big(2-\alpha(s)-\alpha(t)\big)$ in order to have the correct asymptotic behaviour. After symmetrization in s, t, u we have

$$A(s, t, u) = \frac{\bar\beta}{\pi} \left[B\big(1-\alpha(t), 1-\alpha(s)\big) + B\big(1-\alpha(t), 1-\alpha(u)\big) + B\big(1-\alpha(s), 1-\alpha(u)\big)\right] , \tag{3}$$

where we have introduced the Euler B-function

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} .$$

Notice that, in eq. (3), $\bar\beta$ *must* be a constant if we want to have a Regge-like behaviour which, together with crossing, also demands the $1/\big(\Gamma(\alpha)\big)$ t-dependence of the reduced residue function. Equation (3) in fact is hard to modify if one demands an $s^{\alpha-1}$ behaviour in all channels. The only simple generalization of eq. (3) seems to consist in the addition of nonleading and similarly structured terms like $B\big(m-\alpha(t), n-\alpha(s)\big)$ with m, $n \geqslant 1$.

We now discuss some properties of eq. (3) in detail.

1. – Behaviour for large positive s and fixed t.

The first two terms (we shall come to the last one in a moment) give

$$A = \frac{\beta(t)}{\sin \pi\alpha(t)}\left[-\frac{\sin \pi(\alpha(s)+\alpha(t))}{\sin \pi\alpha(s)}\frac{\Gamma(\alpha(s)+\alpha(t)-1)}{\Gamma(\alpha(s))}+\frac{\Gamma(1-\alpha(u))}{\Gamma(2-\alpha(u)-\alpha(t))}\right]. \tag{4}$$

The second term is purely real (for positive s) and goes like $(\alpha(s))^{\alpha(t)-1}$. The first term is the one that corresponds to the Regge term $(-\alpha(s))^{\alpha(t)-1}$ and has both a real and an imaginary part. Some trivial algebra shows that, from the whole eq. (4), we have a real piece like

$$\beta(t)\,\frac{1-\cos \pi\alpha(t)}{\sin \pi\alpha(t)}\,[\alpha(s)]^{\alpha(t)-1}, \qquad \beta(t)=\bar{\beta}/\Gamma(\alpha(t))\,,$$

as in the Regge theory, while the s discontinuity is all contained in the form

$$A \underset{s\to\infty}{\sim} -\beta(t)\,\operatorname{ctg}\alpha(s)\,[\alpha(s)]^{\alpha(t)-1}\,. \tag{5}$$

If $\operatorname{Im}\alpha$ is strictly zero, eq. (5) gives just poles in s and $\operatorname{Im} A$ is a sequence of δ-functions. If $\operatorname{Im}\alpha$ is different from zero and, for instance, increases with s (this happens if the total width does not vary strongly with s), $\operatorname{Im} A$ will describe bumps for moderate values of s, but will finally tend to $\beta(t)(\alpha(s))^{\alpha(t)-1}$ as in the Regge theory (this is due to $\operatorname{ctg}\alpha(s)\to -i$). Of course, the parametrization of eq. (3) can be taken as such only for linearly rising trajectories in which case $(\alpha(s))^{\alpha(t)}$ is equivalent to $(s/s_0)^{\alpha(t)}$. However, we only need a leading term in $\alpha(s)$ going linearly in s, and this does not imply $\operatorname{Im}\alpha=0$. If $\operatorname{Im}\alpha\neq 0$ one probably gets, besides moving poles, other singularities (cuts?) as well.

2. – Singularities in the various channels.

Equation (3) has quite nice analytic features. It has cuts in all the three Mandelstam variables starting from the 2π threshold, where α begins to show an imaginary part. However, if we restrict ourselves to real linear trajectories, our expression has only poles whenever α passes through an integer bigger than 0. Furthermore, because of the $\Gamma(2-\alpha(s)-\alpha(t))$ denominator, no double pole appears, in the sense that the residue in a pole is a polynomial in the other variable.

At first glance our expression shows poles at even values of α as well, in contrast with invariance principles. As these are always nonleading terms, one can in general eliminate them by the addition of nonleading expressions at explained at the beginning. More amusing to notice is the fact that, at least in this reaction, the elimination of spurious singularities can be achieved with a single condition on the trajectory $\alpha(t)$. Take in fact $\alpha(t)=2$. The residue at the pole, produced there by $\Gamma(1-\alpha(t))$ is simply proportional to $\alpha(u)+a(s)$. We then demand

$$\alpha(s)+\alpha(u)=0 \qquad \text{for } \alpha(t)=2, \tag{6}$$

which after some easy algebra gives (always for linear trajectories)

$$\alpha(s)+\alpha(t)+\alpha(u)=2\,. \tag{7}$$

Equation (7) can easily be transformed into the prediction

$$\alpha(-2m_\rho^2+m_\omega^2+3m_\pi^2)=\alpha(-0.53\ (\text{GeV})^2)=0 \tag{8}$$

which was derived in ref. [2] from the sum rules. The reader can verify that eq. (7) is enough to cancel all the undesired poles at even integer values of α. A further interesting consequence of eq. (7) concerns the third term of eq. (3) which could in principle violate the Regge behaviour. Instead, using (7), that term can be rewritten as

$$\frac{\beta(t)}{\sin\pi\alpha(s)}\,\frac{\Gamma(\alpha(s)+\alpha(t)-1)}{\Gamma(\alpha(s))} \tag{9}$$

which is still Regge-behaved. The whole eq. (3) can be rewritten in the form

$$A=\beta(t)\,\frac{\Gamma(\alpha(s)+\alpha(t)-1)}{\Gamma(\alpha(s))}\left[\frac{1-\exp[i\pi\alpha(s)]}{\sin\pi\alpha(s)}+\frac{1-\exp[-i\pi\alpha(t)]}{\sin\pi\alpha(t)}\right], \tag{10}$$

which shows the automatic cancellation of the poles at even integer values of α. By use of (7) one can also write (3) in the very symmetric form

$$A=\frac{\bar\beta}{\pi^2}\,\Gamma(1-\alpha(s))\,\Gamma(1-\alpha(t))\,\Gamma(1-\alpha(u))[\sin\pi\alpha(s)+\sin\pi\alpha(t)+\sin\pi\alpha(u)]. \tag{11}$$

3. – Analysis in terms of Regge poles.

From the asymptotic (Stirling) formula of the Γ-functions in eq. (10) it is seen that our expression corresponds to an infinite family of Regge poles which are equally spaced at all t by two units of angular momentum and with definite relations among their residua. Such a structure of poles was found already both by use of superconvergence [5] and in some recent work [6] based on O_4-symmetry concepts.

4. – Superconvergence sum rules.

Equation (4) has to satisfy all the superconvergence relations. In order to check this in detail we have to introduce a kind of «smoothed» Regge form for Im $A(s,t)$.

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[5] H. R. RUBINSTEIN, A. SCHWIMMER, G. VENEZIANO and M. A. VIRASORO: Weizmann Institute preprint (1968), submitted to *Phys. Rev. Lett.*; see also ref. [3].

[6] G. COSENZA, A. SCIARRINO and M. TOLLER: University of Rome preprint no. 158 and Trieste preprint.

From eq. (4) it can be seen that a form of this kind is

$$\operatorname{Im} A_{\text{Regge}}(s,t) \underset{s\to\infty}{\simeq} \beta(t)\,\frac{\Gamma(\alpha(s)+\alpha(t)-1)}{\Gamma(\alpha(s))}\,. \tag{12}$$

Next we observe that the representation

$$\operatorname{Im} A_{\text{Regge}}(s,t) \underset{s\to\infty}{\simeq} \beta(t)(2\alpha'\nu)^{\alpha(t)-1} = \beta(t)\left[\alpha(s)+\frac{\alpha(t)-2}{2}\right]^{\alpha(t)-1}, \tag{13}$$

where

$$\nu = \frac{s-u}{4}\,, \qquad \alpha(s) = \alpha_0 + \alpha' s\,,$$

is a good approximation of eq. (12) as it coincides also in the next-to-leading term. The form (13) is also the one used in ref. [3].

The first moment sum rule (even moments are trivial) reads then

$$\int_0^{\bar\nu} \nu \operatorname{Im} A(\nu,t)\,\mathrm{d}\nu = \frac{\beta(t)(2\alpha'\bar\nu)^{\alpha(t)-1}\bar\nu^2}{\alpha(t)+1}\,. \tag{14}$$

For positive s the first and third terms of eq. (3) give

$$\operatorname{Im} A(\nu,t) = -\sum_J \frac{\bar\beta\Gamma(1-\alpha(t))}{\alpha'\Gamma(J)\Gamma(2-J-\alpha(t))}\,\delta(s-s_J)(-1)^J + (t\leftrightarrow u)\,. \tag{15}$$

The t, u symmetry has the effect of washing out the poles of even J. If we set $\bar\nu$ midway between the n-th resonance on the leading trajectory and the $(n+1)$-th resonance, eq. (14), after some algebra, reads (writing $\alpha = \alpha(t)$)

$$\alpha\left[1+\sum_{n=1}^{\bar n}\frac{(\alpha+4n)\Gamma(\alpha+2n)}{\Gamma(\alpha+1)\Gamma(2n+1)}\right] = \alpha\,\frac{\Gamma(\alpha+2\bar n+2)}{\Gamma(\alpha+2)\Gamma(2\bar n+1)}\,\Phi_{\bar n+1}(\alpha)\,, \tag{16}$$

where [3]

$$\Phi_{\bar n+1}(\alpha) = \Gamma(2\bar n+1)\Gamma^{-1}(\alpha+2\bar n+2)\left(\frac{\alpha+4\bar n+2}{2}\right)^{\alpha+1}. \tag{17}$$

It is easily seen, by induction, that the two sides of eq. (14) are equal apart from the $\Phi(\alpha)$ function, so that eq. (16) reduces to $\Phi(\alpha)=1$. This condition is very well verified [3] for $|\alpha(t)| < 2\bar n$ and is mathematically true for fixed t and $\bar n \to \infty$ as we wish. If $\alpha(t) \leqslant -1$, νA is superconvergent in the stronger sense of de Alfaro *et al.* [4] and we get from (16)

$$\alpha\left[1+\sum_{n=1}^{\infty}\frac{(\alpha+4n)\Gamma(\alpha+2n)}{\Gamma(\alpha+1)\Gamma(2n+1)}\right] = 0\,, \tag{18}$$

which is just a solution of a superconvergence equation for all t in terms of an infinite number of resonances. If $\alpha > -1$, eq. (16) is a solution of a generalized superconvergence relation [4] and, in this context, the common origin of the two kinds of sum rules is quite clear. In the special case $\alpha = -1$ all but the ϱ contribution vanish and we get $\alpha(t) = \alpha(t)$. Similar considerations can be made for higher-moment sum rules. Note that we have neglected in the right-hand side of eq. (14) the contribution of the u-channel Regge exchange. This approximation is justified by the fact that such a term oscillates in ν as to give an almost zero integral over each range of integration.

5. – Duality of Regge poles and resonances.

It is clearly seen that, in the present model, we have no interference between the Regge term and direct-channel resonances. In a sense, the amplitude is all built up by resonances in such a way that their sum is Regge behaved. In this sense eq. (3) can be considered the inverse of a point of view recently expressed by SCHMID [7] according to which the usual leading Regge term may contain the direct-channel resonances in it.

5. – Possible validity of the Regge formula at subasymptotic energies.

Suppose we have a process whose s-channel quantum numbers are such to prevent any known trajectory to occur. In that case our representation of the amplitude will be given by $\bar{\beta}B(n-\alpha_t(t), m-\alpha_u(u))$ and consequently

$$(19)\qquad A = \frac{\beta(t)}{\sin\pi\alpha(t)}\,\frac{\Gamma(m-\alpha_u(u))}{\Gamma(m+n-\alpha_u(u)-\alpha_t(t))} = \frac{\beta(t)}{\sin\pi\alpha(t)}\,\frac{\Gamma(\alpha_u(s)+c)}{\Gamma(\alpha_u(s)+c-\alpha_t(t)+n)},$$

where c is a constant. Expression (19) has the property of being a real and quite smooth function of s (for positive s). In other words, the absence of resonances in a channel implies for the scattering in the same channel a real Regge term and the validity of the Regge regime at subasymptotic values of the energy. Similar conclusions were reached by use of a different method by HARARI [8].

7. – Predictions for large-t scattering.

Equation (3) predicts the existence of secondary dips and peaks as given by the $\Gamma^{-1}(\alpha(t))$ factor in $\beta(t)$. Furthermore we can draw conclusions for fixed-angle scattering, namely for s, t and u going to infinity at the same time. Letting $x = \cos\theta_s$ and using the Stirling formula we easily get, apart from constant and oscillating factors,

$$(20)\qquad A \underset{\substack{s\to\infty\\ \text{fixed } x}}{\sim} \exp[-f(x)\alpha(s)]\,,\qquad f(x) = \frac{1-x}{2}\log\frac{2}{1-x} + \frac{1+x}{2}\log\frac{2}{1+x}\,.$$

[7] C. SCHMID: *Phys. Rev. Lett.*, **20**, 689 (1968).
[8] H. HARARI: *Phys. Rev. Lett.*, **20**, 1395 (1968).

$f(x)$ is positive for physical angles and turns out to be well represented by $f(x) = \log 2 \cdot \sin\theta_s$. Equation (20) is similar to the form-fitting experiments in pp large-angle scattering [9].

8. – Factorization.

This is perhaps the most delicate point of the discussion. In the scheme proposed here each trajectory is not really an independent object. In most cases (as in $\pi\pi$ scattering) more than one trajectory *must* coexist in a kind of « conspiratorial » situation, namely with certain relations among their trajectory and residue functions, as was found already in the sum rule work [3]. Factorization should be intended here as the existence of a self-consistent set of Regge poles (and relative daughters) whose trajectory functions are independent of the external lines, but consistent in the sense of producing particles at the external-mass values. Furthermore, the reduced residue (defined in a suitable way) should factorize [10]. This is certainly an ambitious and difficult task, which would essentially amount to a complete bootstrap solution of strong interactions.

As a second example let us consider the process $\pi\eta \to \pi\rho$. According to our prescription the invariant amplitude A defined as in eq. (1) will be given by

$$(21)\qquad A_{\pi\eta\to\pi\rho} = \\ = \frac{\bar{\beta}_1}{\pi}\left[B\big(1-\alpha_{A_2}(s), 1-\alpha_\rho(t)\big) + B\big(1-\alpha_{A_2}(u), 1-\alpha_\rho(t)\big) - B\big(1-\alpha_{A_2}(s), 1-\alpha_{A_2}(u)\big)\right],$$

where $\pi\pi \to \eta\rho$ is the t-channel. Imposing to find no poles at even integer value for $\alpha_\rho(t)$ we obtain

$$(22)\qquad \alpha_{A_2}(s) + \alpha_{A_2}(u) + \alpha_\rho(t) = 2\,.$$

Imposing absence of poles at the odd integers for α_{A_2} we find again eq. (22). This demands

$$(23)\qquad \alpha'_\rho = \alpha'_{A_2}\,.$$

Using $m_\rho^2 = 0.6\ (\text{GeV})^2$ and eq. (7) we obtain

$$(24)\qquad \alpha_{A_2}(0) = 1 - \frac{1}{2}\,\frac{3m_\rho^2 - m_\omega^2 - m_\pi^2 + m_\eta^2}{3m_\rho^2 - m_\omega^2 - 3m_\pi^2} \simeq 0.36\,.$$

We thus predict $m_{A_2} \simeq 1350$ MeV.

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(9) G. Cocconi, V. T. Cocconi, A. D. Krisch, J. Orear, R. Rubinstein, D. B. Searl, B. T. Ulrich, W. F. Baker, E. W. Jenkins and A. L. Read: *Phys. Rev.*, **138**, B 165 (1965). Having linearly rising trajectories we are also consistent with the Cerulus-Martin bound. See C. B. Chiu and C. I. Tan: *Phys. Rev.*, **162**, 1701 (1967).

(10) We know that, at $t = 0$, a simple (irreducible) Lorentz pole does obey factorization (see ref. (1)). It seems also plausible to conjecture (M. Toller: private communication) that this is the only case in which factorization is fulfilled. Since our expression does not probably correspond to a single Lorentz pole, nonleading terms might be needed in order to have factorization. We thank M. Toller for discussion on this point.

As a third example one could try to build up a scattering amplitude for $s+s \to s+s$ (s being a scalar particle with the vacuum quantum numbers) and try to ask dominance of a leading trajectory passing by the particle itself. This is seen to be impossible with a positive slope of α, since the equation similar to (7) gives $\alpha(0) = 1$.

It is possible to extend the above considerations to the more interesting case of $\pi\pi$ *scattering* and to obtain a crossing symmetric amplitude in the approximation of ρ and f trajectory dominance and disregarding the Pomeranchuk contribution, according to a now accepted philosophy [8,11]. We find consistency only if $\alpha_\rho = \alpha_f = \alpha$ and $\alpha(0) \simeq \frac{1}{3}$. Furthermore, we can predict $\pi\pi$ scattering lengths in terms of $g^2_{\rho\pi\pi}$ and obtain (apart from the Pomeranchuk contribution)

$$a_0 = \tfrac{5}{2} \cdot a_2 \simeq -1.25\, m_\pi^{-1} \,.$$

Further details as well as applications of this scheme to more complicated cases will be considered elsewhere.

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(11) It may be, however, that one runs into difficulties in adding the Pomeranchuk contribution at the end in a crossing-symmetric way. An alternative interesting possibility would be to consider it as originated somehow by the other trajectories (through their nonresonating parts) and not as an independent object. This problem, which certainly requires further study, is closely connected to that of the nature of the Pomeranchuk singularity.