Beyond Unitarity: New on-shell representations for loop amplitudes

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(based on 1509.02169,
with: Baadsgaard, Bjerrum-Bohr, Bourjaily, Damgaard&Feng)

- In going to Solvay 1963, Feynman wondered how to explain quantum electrodynamics to 1913 physicists
- He realized they understood ‘vacuum energy’
- So he conceived of two boxes, one with a gas of hydrogen atoms in $2S_{1/2}$ state, the other in $2P_{1/2}$
- Photons in these two boxes would have different refractive index, hence different vacuum energies
- This would be interpreted as a contribution to E_{2P}-E_{2S} : *the Lamb shift*

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- Mathematically, photon dispersion relation measures its forward amplitude against atoms
- To generalize: use $\hbar\omega/2$ to normalize fluctuations

$$\mathcal{A}^{(1)}(\{p_i\}) = \sum_\lambda \frac{1}{2}(-1)^F \int \frac{d^3\ell}{(2\pi)^3 2E_\ell} \mathcal{A}^{\text{tree}}(\ell_\lambda, -\ell_{-\lambda}, \{p_i\})$$

- From this, Feynman derived (for the first time) the Faddev-Popov ghost in Yang-Mills&gravity [@one loop]
- This was used in early string theory, in a proof of no-ghost theorem, helping establish the rules

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- Problems: Forward limits generally singular; tree theorem doesn't fully extent to higher-loops

  [Catani,Bierenbaum et al '10...; SCH '10]

- In planar case, bypassed by loop integrand recursion

  [Arkani-Hamed,Bourjaily, Cachazo,SCH&Trnka,'10]

- I'll present new hybrid representations, combining features of unitarity-based methods

  -Express loops as integrals over on-shell trees
  -Manifestly ghost-free
  -Valid in any quantum field theory
  -Can be integrated termwise with standard methods

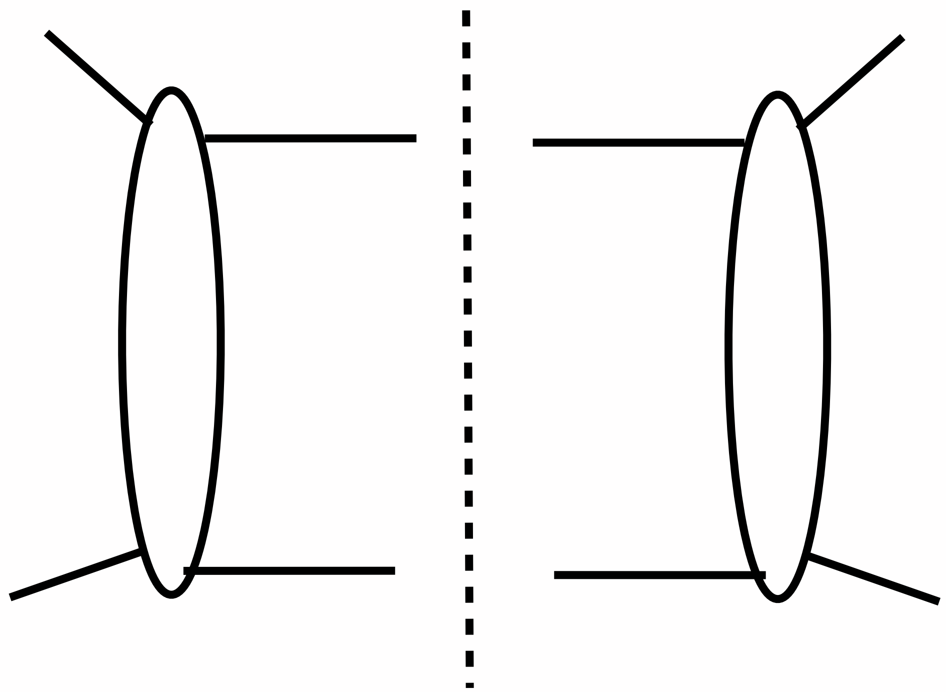
Outline

1. Introduction
2. Context
   -Unitarity method and amplitude calculations
   -Scattering equations
3. Three questions addressed:
   -How to integrate expressions term-wise?
   -How to make sense of forward limits
   -How to extend to higher loops?
4. Conclusions

Context

- Progress in precision calculations of scattering amplitudes: spurred by collider applications and fundamental desire to understand structure
- Modern attitude is generally inverse to the tree theorem: to go back to the trees
- Driven by simplicity of trees in gauge theories (compared to Feynman rules expansion)

Unitarity method



Cut diagram
=
Product of trees

- Match all cuts to solve for the integrand

$$\mathcal{A}^{(L)} = \sum_k c_k \int_\ell \mathcal{I}_k^{(L)}$$

products of trees

standardized integral basis

[Bern,Dixon,Kosower &Dunbar'95-...]

- Unitarity method:

$$\mathcal{A}^{(L)} = \sum_k c_k \int_\ell \mathcal{I}_k^{(L)}$$

M^{-1} . trees

- Generally 'all or nothing': until all c_k are found, little information is gained (e.g. limits hard to extract)

- Relations to trees in this talk will be fully explicit:

$$\mathcal{A}^{(L)} = \int_\ell \mathcal{I}^{(L)}(\tilde{\ell}(\ell))$$

products of trees

evaluated at shifted arguments

Partial fractions

$$\frac{1}{ab} = \frac{1}{b-a}\left[\frac{1}{a} - \frac{1}{b}\right]$$

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- Really: partial fraction in η^2 where $\ell^2 \mapsto \ell^2 + \eta^2$

$$\left.\frac{1}{\ell^2+\eta^2}\,\frac{1}{(\ell+p_1)^2+\eta^2}\right|_{\eta^2=0} = \frac{1}{2\ell\cdot p_1 + p_1^2}\left[\frac{1}{\ell^2} - \frac{1}{(\ell+p_1)^2}\right]$$

$$\frac{1}{D_1\cdots D_m} = \sum_j \frac{1}{D_j}\prod_{k\neq j}\frac{1}{D_k - D_j}$$

More on partial fractions

- Feynman's proof of his tree theorem, amounts to partial fraction in energy
- BCFW's proof of recursion relation is partial fraction in z $(\text{with } A(z) = A(\ell_1{+}zq, \ell_2{-}zq, \ldots))$
  [Britto,Cachazo,Feng&Witten'05]
- Here, we partial-fraction in η^2: $\ell \mapsto \ell{+}\eta$ extra-dimensional component of loop momenta (perpendicular to all D=4-2ε)
  -*Works in any theory where dim.reg. is used*

Inspiration *or brief history...*

- Recent novel representation of trees, localized on zeros of 'scattering equations': [Cachazo,He&Huang, `13]

$$\sum_k \frac{p_j \cdot p_k}{z_j - z_k} = 0$$

- Shortly derived from 'ambitwistor string': [Mason&Skinner '13;...]

$$S = \frac{1}{2\pi} \int \left(\eta^{\mu\nu} P_\mu \bar{\partial} X_\nu - \frac{1}{2} e \eta^{\mu\nu} P_\mu P_\nu \right) + \dots$$

- Extended to loop-level, led to complicated elliptic functions

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- Extended to loop-level, led to complicated elliptic functions [Adamo,Casali&Skinner '13;...]

- Moduli integral could be localized on boundary

[Geyer,Mason,Monteiro,Tourkine, `15]

- [Reminiscent of a proof of no-ghost theorem?]

[Brink&Olive, '73]

- General structure: $1/\ell^2 \times$ [linear denominators]

$$\hat{\mathcal{M}}_5^{(1)} = \frac{1}{32\,\ell^2} \sum_{\sigma \in S_5} \frac{1}{\prod_{i=1}^4 \left(\ell \cdot \sum_{j=1}^i k_{\sigma_i} + \frac{1}{2}(\sum_{j=1}^i k_{\sigma_i})^2\right)} \times \ldots$$

- Partial fractions used to compare with known result

- Connection with forward amplitudes, made on-shell via η^2 shift, explicitly made in ϕ^3 case

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[He&Huang, '15]

Questions

1. How to integrate expressions term-wise? (Contour?)

2. How to make sense of forward limits?

3. How to extend to higher loops?

Integration contour

- Q1. Can individual terms be integrated?

- Earlier attitude, for BCFW loop recursion:
  No good technique: before integration one must line-up and cancel spurious denominators
  [Arkani-Hamed,Bourjaily,Cachazo,SCH&Trnka '10]

- Reinforced by calculation of MHV 1-loop on $R^{1,3}$
  [Lipstein&Mason '13]

Contour derivation

- Start from a Feynman integral:

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- Important: the (positive) ε_j can have any rel. size
- General denominator factor after partial fractions:

$$\frac{1}{D_i - D_j + i(\epsilon_i - \epsilon_j)}$$

- Individual term will depend on ε ordering

[though the sum will not]

- Options:
  -Fix some arbitrary ordering [difficult beyond planar]
  -Average over all choices!
- Example: three denominators. 3!=6 orderings

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$$\frac{1}{D_1+i\epsilon}\frac{1}{D_{21}+i(\epsilon_2-\epsilon_1)}\frac{1}{D_{31}+i(\epsilon_3-\epsilon_1)}$$

$$=\frac{1}{6}\frac{1}{D_1+i\epsilon}\left[\frac{2}{(D_{21}+i\epsilon)(D_{31}+i\epsilon)}+\frac{1}{(D_{21}-i\epsilon)(D_{31}+i\epsilon)}\right.$$
$$\left.+\frac{1}{(D_{21}+i\epsilon)(D_{31}-i\epsilon)}+\frac{2}{(D_{21}-i\epsilon)(D_{31}-i\epsilon)}\right]$$

$$=\frac{1}{D_1+i\epsilon}\left[\mathcal{P}\frac{1}{D_{21}}\mathcal{P}\frac{1}{D_{31}}-\frac{\pi^2}{3}\delta(D_{21})\delta(D_{31})\right]$$

(used: $\frac{1}{x+i\epsilon}=\mathcal{P}\frac{1}{x}-i\pi\delta(x)$)

- Looks weird, but standard techniques work
- Example: Schwinger parameters

$$\frac{i}{x+i\epsilon} \Leftrightarrow \int_0^\infty da e^{iax}, \quad \mathcal{P}\frac{2i}{x} \Leftrightarrow \int_{-\infty}^{\infty} da\,\mathrm{sign}(a) e^{iax}, \quad 2\pi\delta(x) \Leftrightarrow \int_{-\infty}^{\infty} da e^{iax}$$

Try bubble:

$$\int \frac{d^d\ell}{\pi^{d/2}} \frac{1}{\ell^2+i\epsilon} \mathcal{P}\frac{2}{2\ell\cdot p+p^2} \Leftrightarrow \Gamma(d-2)\int_{-\infty}^{\infty} da \frac{\mathrm{sign}(a)}{(-a(1-a)p^2 - i\epsilon)^{2-d/2}}$$

a<0 and a>1 cancel each other, leaving usual result

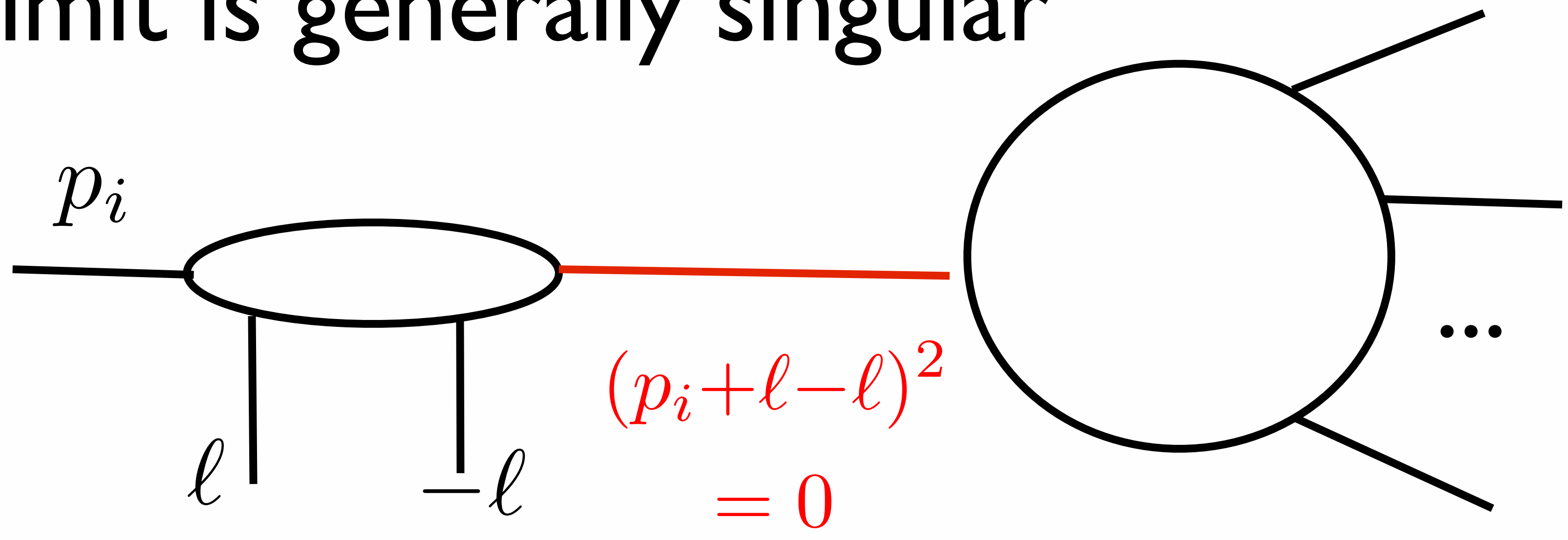
- Higher-point Schwinger parameters reproduced through amusing identities:

$$\theta(a+b+c)(\mathrm{sign}(a)\mathrm{sign}(b) + \mathrm{sign}(a)\mathrm{sign}(c) + \mathrm{sign}(b)\mathrm{sign}(c) + 1) = \theta(a)\theta(b)\theta(c)$$

- Lesson: terms can be integrated separately, on a simple contour ('average ε'), so that the sum reproduces original integral
- Schwinger parameters work well
- Integration-by-part identities work as usual
- [Nontrivial contour reminiscent of tree theorem]

Forward limits

- Q2. How to make sense of forward limits?
- Limit is generally singular



- Well-defined in SUSY, due to cancelations
  (related to solution of hierarchy problem)

[SCH '10;...; Benincasa '15]

- General solution: partial-fractions! Start from:

$$I(\ell) = \frac{1}{\ell^2}\left[\frac{N(\ell)}{(2\ell\cdot P_1+Q_1^2)\cdots(2\ell\cdot P_m+Q_m^2)}\right] \equiv \frac{1}{\ell^2}\tilde{I}(\ell)$$

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- Three types of poles:

-α=0:

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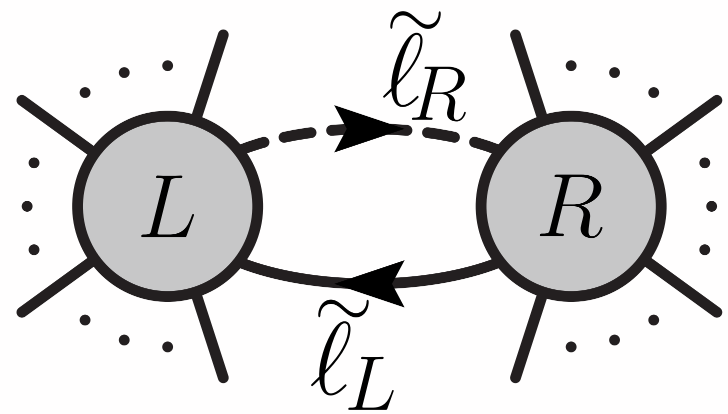
-α=finite: physical unitarity cut

- Up to vanishing integrals, forward limit okay!

Q-cut representation



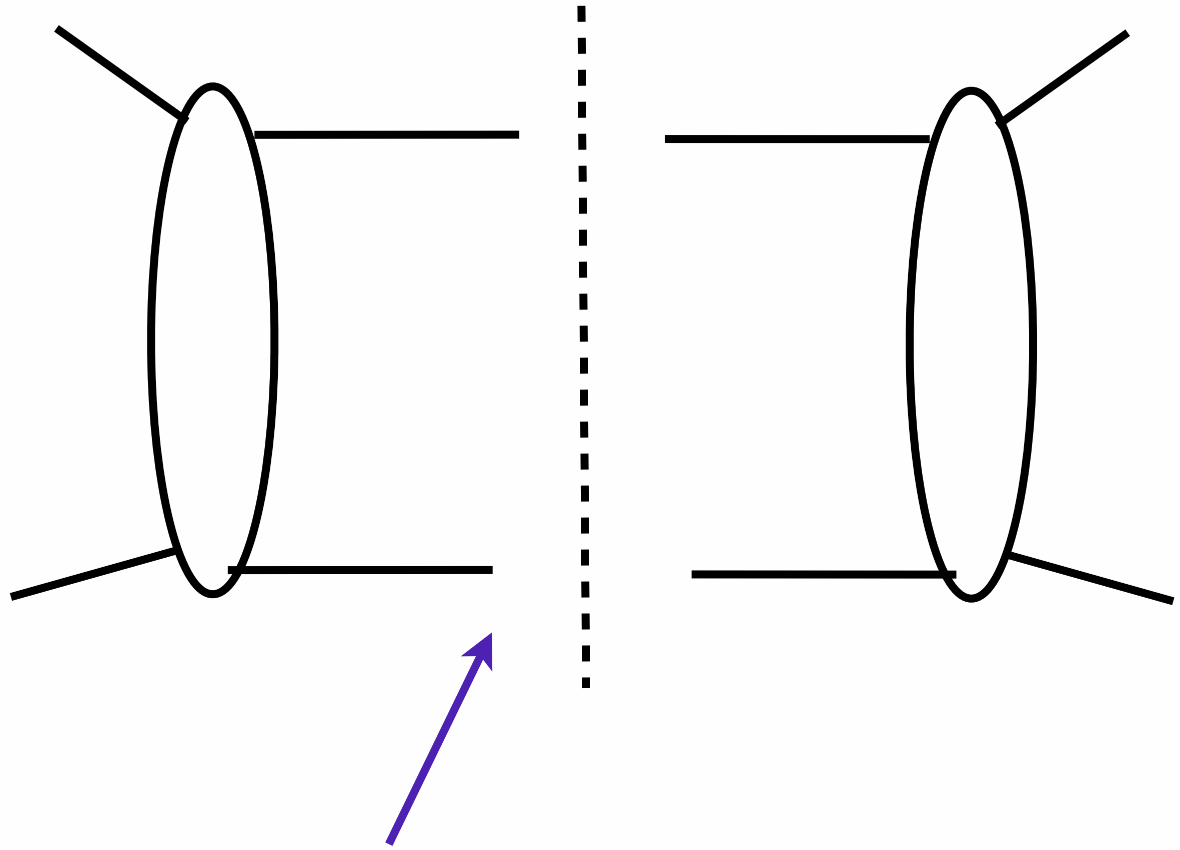
$$\mathcal{A}^{(1)} = \sum \int \frac{d^d\ell}{(2\pi)^d} \mathcal{A}_L(\cdots, \widetilde{\ell}_L, -\widetilde{\ell}_R) \frac{1}{\ell^2} \frac{1}{(2\ell\cdot P_L + P_L^2)} \mathcal{A}_R(\widetilde{\ell}_R, -\widetilde{\ell}_L, \cdots),$$

$$\text{where}: \quad \tilde{\ell}_L = \alpha(\ell+\eta), \quad \tilde{\ell}_R = \tilde{L} + P_L$$

$$\text{with: } \eta^2 = -\ell^2, \quad \alpha = -P_L^2/(2\ell\cdot P_L).$$

$$\Rightarrow \tilde{\ell}_L^2 = 0 = \tilde{\ell}_R^2$$

No forward limit: trees are nonsingular and well-defined



Cut can include either:
-Physical states
-Physical states + Unphysical states + ghosts

Ghosts unnecessary

Higher loops

- Q3. How to extend to higher loops?
- For tree theorem, most obvious guess: couple two independent vacuum fluctuations

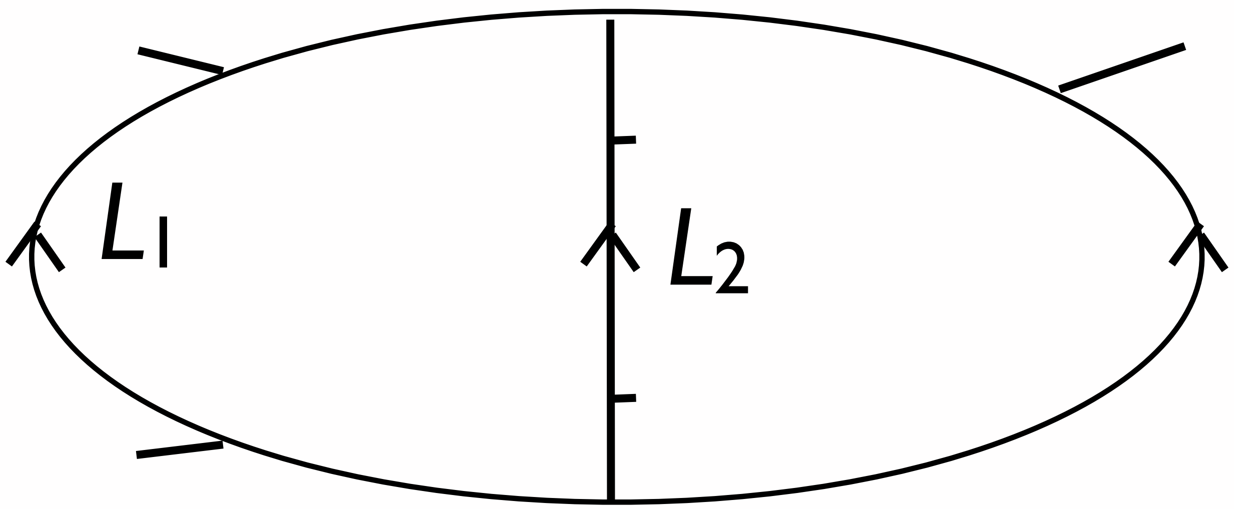
$$\mathcal{A}^{(2)}(\{p_i\}) \sim \frac{1}{4}\int \frac{d^3\ell_1}{(2\pi)^3 2E_{\ell_1}} \frac{d^3\ell_2}{(2\pi)^3 2E_{\ell_2}} \mathcal{A}^{\text{tree}}(\ell_1, -\ell_1, \ell_2, -\ell_2, \{p_i\})$$

- Wrong guess: too 'classical'; no stable vacuum ('ultraviolet catastrophe'); unitarity violated

[Holdom; SCH '10]

A no-go

- At 2-loop: first pick two preferred cycles



- Using clever contour integration, cutting-open two lines is not a problem [Feynman '63,'69; Catani, Bierenbaum et al; SCH'10]
- The issue is that the result depends on orientation between L_1 and L_2.

- Causes problems when collecting graphs

$$\mathcal{A}^{(2)}(\{p_i\}) \sim \frac{1}{4}\int \frac{d^3\ell_1}{(2\pi)^3 2E_{\ell_1}} \frac{d^3\ell_2}{(2\pi)^3 2E_{\ell_2}} \text{“}\mathcal{A}\text{”}(\ell_1, -\ell_1, \ell_2, -\ell_2, \{p_i\}$$

- One finds that “*A*” must distinguish graphs where:

$$\begin{cases} \text{-a } (L_1+L_2+P)^2 \text{ propagator appears} \\ \text{-a } (L_1-L_2+P)^2 \text{ propagator appears} \\ \text{-none of the above} \end{cases}$$

- This prevents “*A*” to be physically meaningful (even gauge invariant) and was recognized by Feynman
- In 2010, I marginally improved, by noting that in the planar limit one can use color flow as a proxy

two-loop Q-cuts

- In previous graph, use three-parameter deformation:

$$\ell_{1,2} \mapsto \ell_{1,2} + \eta_{1,2}$$

$$\text{where } \eta_1^2 = z_1, \quad \eta_2^2 = z_2, \quad (\eta_1+\eta_2)^2 = z_3.$$

- The η's (as before) are orthogonal to all external and D=4-2eps loop variables

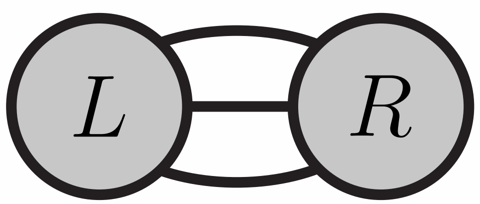
  This is possible: within dim.reg. the space is effectively infinite dimensional (no linear relations among loop momenta)

- With three-parameters, Feynman's no-go avoided!

two-loop Q-cuts

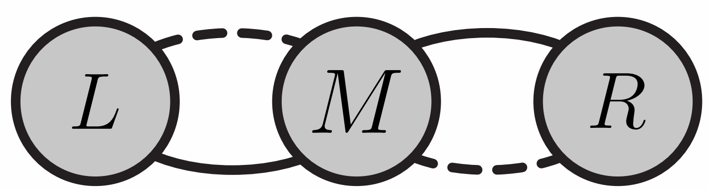
- Doing partial-fractions separately in z_1, z_2, z_3, yields two nonvanishing contributions:

1. All z_i finite: (three propagators cut)



$$\Leftrightarrow \frac{\mathcal{A}_L(\widetilde{\ell}_1, \widetilde{\ell}_2, -\widetilde{\ell}_3)\mathcal{A}_R(-\widetilde{\ell}_1, -\widetilde{\ell}_2, \widetilde{\ell}_3)}{\ell_1^2 \ell_2^2 (\ell_1 + \ell_2 + P_L)^2}$$

2. Two z_i finite, one at infinity: two cut propagators. Then apply two α-deformations:

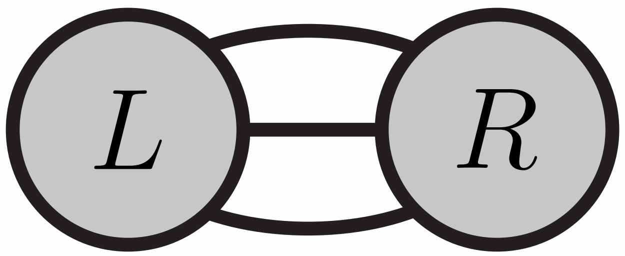


$$\Leftrightarrow \qquad (19)$$

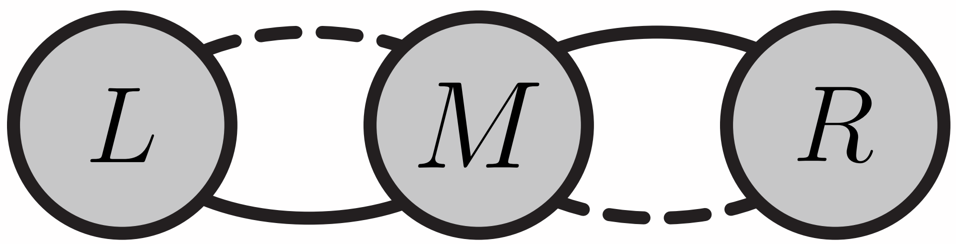
$$\frac{\mathcal{A}_L(\widetilde{\ell}_1, -\widetilde{\ell}_1 - P_L)}{\ell_1^2 (2\ell_1 \cdot P_L + P_L^2)} \mathcal{A}_M(-\widetilde{\ell}_1, \widetilde{\ell}_1 + P_L, -\widetilde{\ell}_2, \widetilde{\ell}_2 + P_R) \frac{\mathcal{A}_R(\widetilde{\ell}_2, -\widetilde{\ell}_2 - P_R)}{\ell_2^2 (2\ell_2 \cdot P_R + P_R^2)}$$

- Thus

$$\mathcal{A}^{(2)} = \sum \int \frac{d^d \ell_1}{(2\pi)^d} \frac{d^d \ell_2}{(2\pi)^d} \Bigg[$$



$$+$$



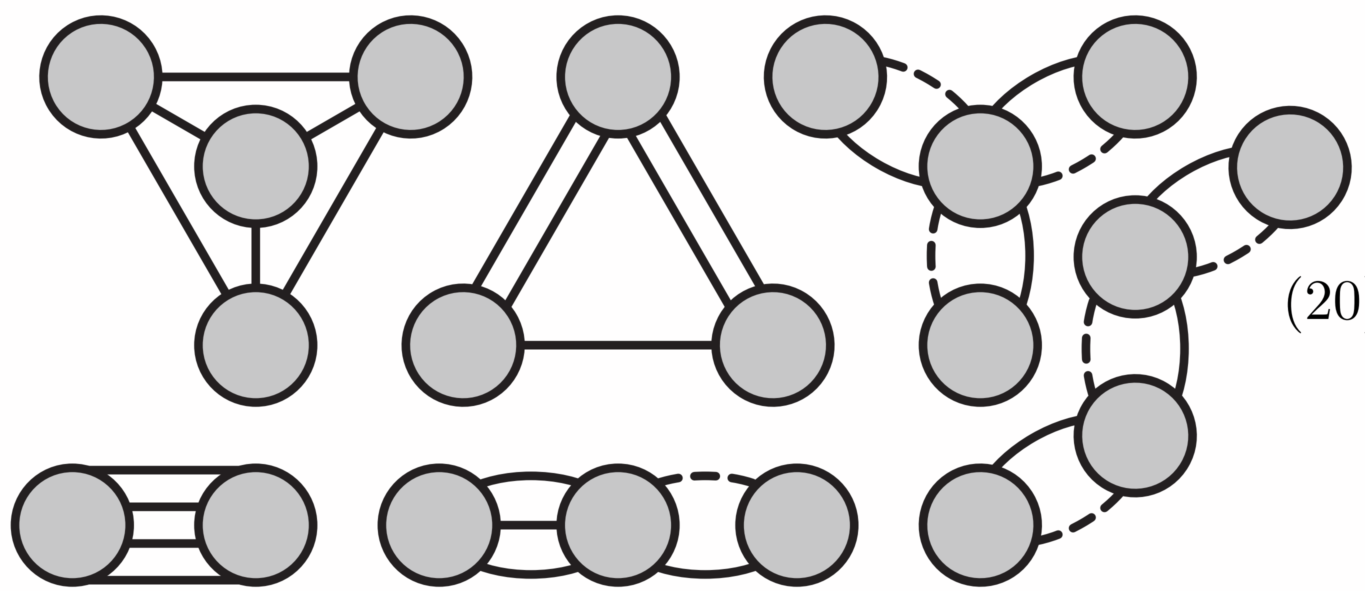
$$\Bigg]$$

- Contour for individual term can be derived and is similar as before
- Gauge-invariant separation between the two topologies achieved by $z_i \to \infty$ projections

3-loops

- At 3-loops, the three momenta and their dot products allow for 6-variables deformation
- Including all subtopologies (residues at infinity):

$$\mathcal{A}^{(3)} = \sum$$



(20)

- A nice theorem: Q-cut integrands depend only on four-dimensional part of *L* (proof: amplitude themselves depend only on $\ell_\perp^2 + \eta^2$)

- That is,

$$\int \frac{d^4\ell}{(2\pi)^4} \int \frac{d^{-2\epsilon}\ell_\perp}{(2\pi)^{-2\epsilon}} \frac{1}{\ell^2 - \ell_\perp^2 + i\epsilon} \tilde{I}(\ell)$$

$$= c \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{(-\ell^2 - i\epsilon)^{1+\epsilon}} \tilde{I}(\ell)$$

gauge-invariant 4D function

- Taking this as the starting point, any 4D regulator will also work!

Outlook

Q-cuts: $\mathcal{A}^{(L)} = \int d^d\ell_1 \cdots d^d\ell_L \; [\text{products of on-shell trees}]$

- Solves outstanding problem: express loops from on-shell trees
- New ingredient: linear denominators
- Conceptually clean starting point for:
  -Applications at two-loop and higher
  -Study of limits using on-shell methods
  -Taking *D*=4 limit
  -New regulators (gauge-invariance built-in!)
  -...

When would we be able to explain the steps of a two-loop QCD cross-section calculation, to a 191(3)3 physicist?