Hidden symmetries of scattering amplitudes:

from $\mathcal{N} = 4$ SYM to QCD

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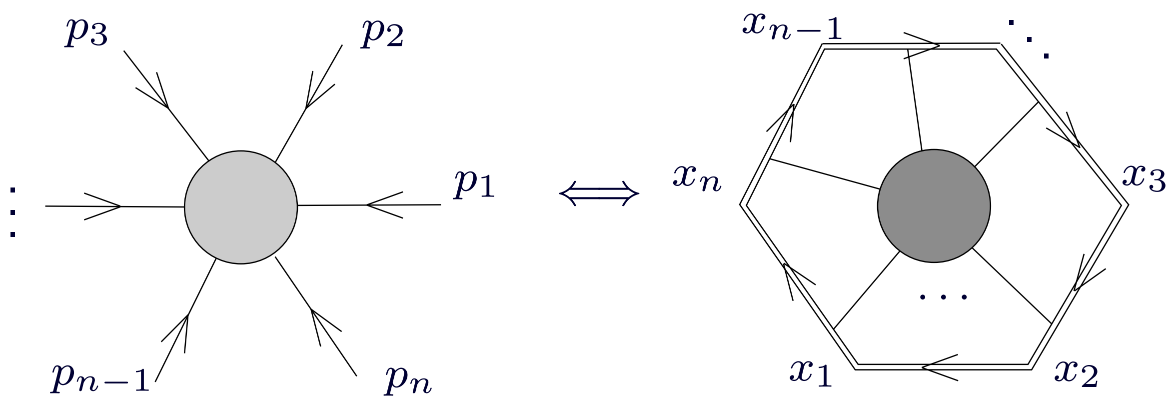
- On-shell gluon scattering amplitudes
- Iterative structure at weak/strong coupling in $\mathcal{N}=4$ SYM
- Dual conformal invariance – hidden symmetry of planar amplitudes
- Scattering amplitude/Wilson loop duality in $\mathcal{N}=4$ SYM
- Scattering amplitude/Wilson loop duality in QCD



$\langle 0|S|1^-2^-3^+\dots n^+\rangle$ $\qquad\qquad$ $\langle 0|\,\mathrm{tr}\,\mathrm{P}\exp\left(i\oint_C dx\cdot A(x)\right)|0\rangle$

Why $\mathcal{N}=4$ super Yang-Mills theory is interesting?

- ✔ Four-dimensional gauge theory with extended spectrum of physical states/symmetries

  2 gluons with helicity $\pm 1\,,$ *6 scalars with helicity* $0\,,$ *8 gaugino with helicity* $\pm \frac{1}{2}$

  all in the adjoint of the $SU(N_c)$ gauge group

- ✔ All classical symmetries survive at quantum level:
  - ✗ Beta-function vanishes to all loops $\Longrightarrow$ the theory is (super)conformal
  - ✗ The theory contains only two free parameters: 't Hooft coupling constant $\lambda = g^2_{\rm YM} N_c$ and the number of colors N_c

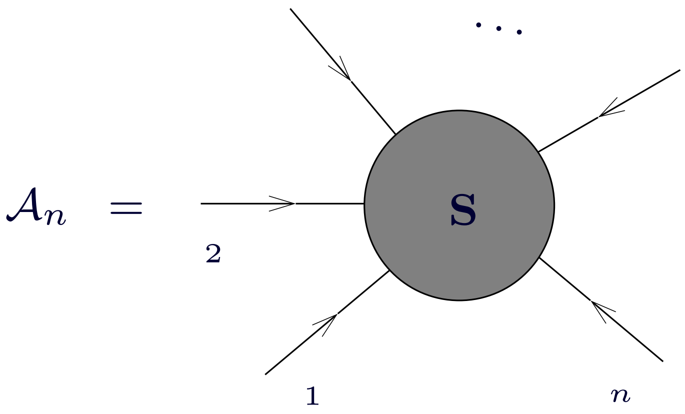
- ✔ Why $\mathcal{N}=4$ SYM theory is fascinating?
  - ✗ *At weak coupling*, the number of contributing Feynman integrals is *MUCH* bigger compared to QCD ... but the final answer is *MUCH* simpler (examples to follow)
  - ✗ *At strong coupling*, the conjectured AdS/CFT correspondence [Maldacena],[Gubser,Klebanov,Polyakov],[Witten]

    Strongly coupled planar $\mathcal{N}=4$ *SYM* $\Longleftrightarrow$ *Weakly coupled string theory on* $\mathrm{AdS}_5 \times \mathrm{S}^5$

  - ✗ Final goal (dream):

$\mathcal{N}=4$ SYM theory is a unique example of the four-dimensional gauge theory that can be/ should be/ would be solved exactly for arbitrary value of the coupling constant!!!

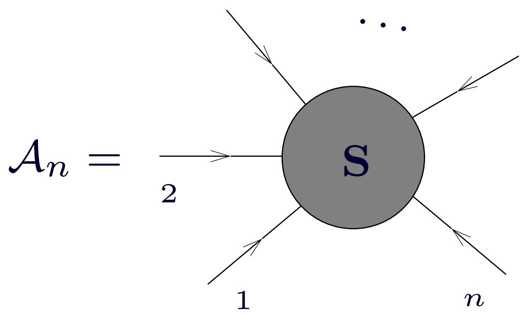
Why scattering amplitudes?



- On-shell matrix elements of $S-$matrix:
  - Probe (hidden) symmetries of gauge theory
  - Are independent on gauge choice
  - Nontrivial functions of Mandelstam variables $s_{ij} = (p_i + p_j)^2$
- Simpler than QCD amplitudes but they share many of the same properties
- In planar $\mathcal{N} = 4$ SYM theory they seem to have a remarkable structure
- All-order conjectures and a proposal for strong coupling via AdS/CFT
- Hints for new symmetry – dual superconformal invariance

On-shell gluon scattering amplitudes in $\mathcal{N} = 4$ SYM

✔ Gluon scattering amplitudes in $\mathcal{N} = 4$ SYM



✗ Quantum numbers of on-shell gluons $|i\rangle = |p_i, h_i, a_i\rangle$: momentum ($(p_i^\mu)^2 = 0$), helicity $(h = \pm 1)$, color (a)

✗ Suffer from IR divergences $\mapsto$ require IR regularization

✗ Close cousin to QCD gluon amplitudes

✔ Color-ordered planar partial amplitudes

$$\mathcal{A}_n = \mathrm{tr}\left[T^{a_1}T^{a_2}\dots T^{a_n}\right] A_n^{h_1,h_2,\dots,h_n}(p_1, p_2, \dots, p_n) + \text{[Bose symmetry]}$$

✗ Color-ordered amplitudes are classified according to their helicity content $h_i = \pm 1$

✗ Supersymmetry relations:

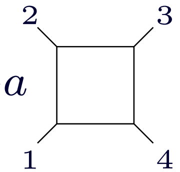
$$A^{++\dots+} = A^{-+\dots+} = 0\,, \qquad A^{(\mathrm{MHV})} = A_n^{--+\dots+}\,, \qquad A^{(\mathrm{next-MHV})} = A_n^{---+\dots+}\,, \quad \dots$$

✗ The $n = 4$ and $n = 5$ planar gluon amplitudes are all MHV

✗ *Weak/strong coupling corrections to all MHV amplitudes are described by a single function of 't Hooft coupling and kinematical invariants!* [Parke,Taylor]

$$A_n^{\mathrm{MHV}} = \delta(p_1 + \dots + p_n) A_n^{(\mathrm{tree})}(p_i, h_i) M_n^{\mathrm{MHV}}(\{s_{i,i+1}\}; \lambda)$$

Four-gluon amplitude in $\mathcal{N}=4$ SYM at weak coupling

$$\mathcal{A}_4/\mathcal{A}_4^{(\text{tree})} = 1 + a$$ 

$$+ O(a^2)\,, \qquad a = \frac{g_{\text{YM}}^2 N_c}{8\pi^2}$$

[Green,Schwarz,Brink'82]

All-loop planar amplitude can be split into a IR divergent and a finite part

$$\mathcal{A}_4(s,t) = \text{Div}(s,t,\epsilon_{\text{IR}})\,\text{Fin}(s/t)$$

- ✔ IR divergences appear to all loops as poles in ϵ_{IR} (in dim.reg. with $D = 4 - 2\epsilon_{\text{IR}}$)
- ✔ IR divergences exponentiate (in any gauge theory!) [Mueller],[Sen],[Collins],[Sterman],[GK]'78-86

$$\text{Div}(s,t,\epsilon_{\text{IR}}) = \exp\left\{-\frac{1}{2}\sum_{l=1}^{\infty} a^l \left(\frac{\Gamma_{\text{cusp}}^{(l)}}{(l\epsilon_{\text{IR}})^2} + \frac{G^{(l)}}{l\epsilon_{\text{IR}}}\right)\left[(-s)^{l\epsilon_{\text{IR}}} + (-t)^{l\epsilon_{\text{IR}}}\right]\right\}$$

- ✔ *IR divergences* are in the one-to-one correspondence with *UV divergences* of Wilson loops

[Ivanov,GK,Radyushkin'86]

$$\Gamma_{\text{cusp}}(a) = \sum_l a^l \Gamma_{\text{cusp}}^{(l)} = \textit{cusp anomalous dimension of Wilson loops}$$

$$G(a) = \sum_l a^l G_{\text{cusp}}^{(l)} = \textit{collinear anomalous dimension}$$

- ✔ *What about finite part of the amplitude* $\text{Fin}(s/t)$*? Does it have a simple structure?*

$$\text{Fin}_{\text{QCD}}(s/t) = [\text{4 pages long mess}]\,, \qquad \text{Fin}_{\mathcal{N}=4}(s/t) = \text{BDS conjecture}$$

Four-gluon amplitude in $\mathcal{N}=4$ SYM at weak coupling II

- ✔ Bern-Dixon-Smirnov (BDS) conjecture:

$$\text{Fin}(s/t) = 1 + a\left[\tfrac{1}{2}\ln^2(s/t) + 4\zeta_2\right] + O(a^2) \overset{\text{all loops}}{\Longrightarrow} \exp\left[\frac{\Gamma_{\text{cusp}}(a)}{4}\ln^2(s/t) + \text{const}\right]$$

  - ✗ Compared to QCD,

    (i) the complicated functions of s/t are replaced by the elementary function $\ln^2(s/t)$;

    (ii) no higher powers of logs appear in $\ln(\text{Fin}(s/t))$ at higher loops;

    (iii) the coefficient of $\ln^2(s/t)$ is determined by the cusp anomalous dimension $\Gamma_{\text{cusp}}(a)$!?

  - ✗ The conjecture has been verified up to three loops [Anastasiou,Bern,Dixon,Kosower'03],[Bern,Dixon,Smirnov'05]
  - ✗ A similar conjecture exists for n-gluon MHV amplitudes [Bern,Dixon,Smirnov'05]
  - ✗ It has been confirmed for $n=5$ at two loops [Cachazo,Spradlin,Volovich'04], [Bern,Czakon,Kosower,Roiban,Smirnov'06]
  - ✗ Agrees with the strong coupling prediction from the AdS/CFT correspondence [Alday,Maldacena'06]

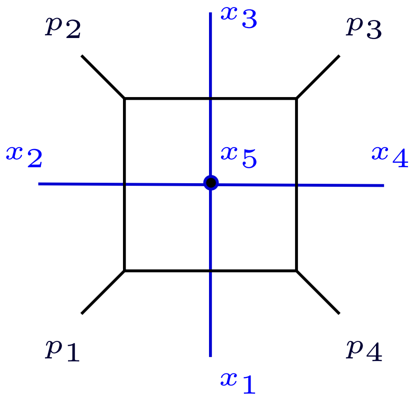
- ✔ *Surprising features of the finite part of the MHV amplitudes in planar $\mathcal{N}=4$ SYM:*
  - ☞ Why should finite corrections exponentiate?
  - ☞ Why should they be related to the cusp anomaly of Wilson loop?

Dual conformal symmetry

Examine one-loop 'scalar box' diagram

- ✔ Change variables to go to a *dual 'coordinate space'* picture (not a Fourier transform!)

$$p_1 = x_1 - x_2 \equiv x_{12}\,, \quad p_2 = x_{23}\,, \quad p_3 = x_{34}\,, \quad p_4 = x_{41}\,, \quad k = x_{15}$$



$$= \int \frac{d^4k\,(p_1+p_2)^2(p_2+p_3)^2}{k^2(k-p_1)^2(k-p_1-p_2)^2(k+p_4)^2} = \int \frac{d^4x_5\; x_{13}^2 x_{24}^2}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}$$

Check conformal invariance by inversion $x_i^\mu \to x_i^\mu / x_i^2$

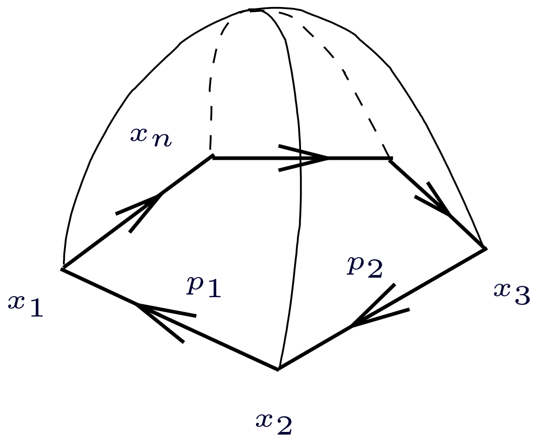
[Broadhurst],[Drummond,Henn,Smirnov,Sokatchev]

- ✔ The integral is invariant under conformal $SO(2,4)$ transformations in the dual space!
- ✔ The symmetry *is not related* to conformal $SO(2,4)$ symmetry of $\mathcal{N}=4$ SYM
- ✔ All scalar integrals contributing to A_4 up to four loops possess the dual conformal invariance!
- ✔ The dual conformal symmetry allows us to determine four- and five-gluon planar scattering amplitudes to all loops! [Drummond,Henn,GK,Sokatchev],[Alday,Maldacena]
- ✔ Dual conformality is slightly broken by the infrared regulator
- ✔ For *planar* integrals only!

Four-gluon amplitude from AdS/CFT

Alday-Maldacena proposal:

✔ On-shell scattering amplitude is described by a classical string world-sheet in AdS_5



✗ On-shell gluon momenta $p_1^\mu, \ldots, p_n^\mu$ define sequence of light-like segments on the boundary

✗ The closed contour has n cusps with the *dual coordinates* x_i^μ (the same as at weak coupling!)

$$x_{i,i+1}^\mu \equiv x_i^\mu - x_{i+1}^\mu := p_i^\mu$$

The dual conformal symmetry also exists at strong coupling!

✔ Is in agreement with the Bern-Dixon-Smirnov (BDS) ansatz for $n = 4$ amplitudes

✔ Admits generalization to arbitrary $n-$gluon amplitudes but it is difficult to construct explicit solutions for 'minimal surface' in AdS

✔ Agreement with the BDS ansatz is also observed for $n = 5$ gluon amplitudes [Komargodski] but disagreement is found for $n \to \infty$ $\mapsto$ *the BDS ansatz needs to be modified* [Alday,Maldacena]

The same questions to answer as at weak coupling:

☞ *Why should finite corrections exponentiate?*

☞ *Why should they be related to the cusp anomaly of Wilson loop?*

From gluon amplitudes to Wilson loops

Common properties of gluon scattering amplitudes at both weak and strong coupling:

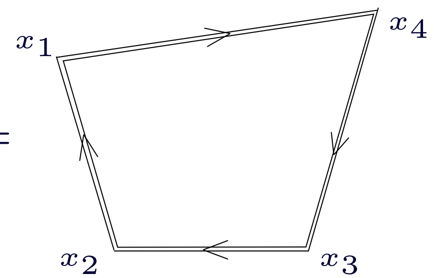
(1) IR divergences of $\mathcal{A}_4$ are in one-to-one correspondence with UV div. of *cusped Wilson loops*

(2) The gluons scattering amplitudes possess a hidden *dual conformal symmetry*

☞ *Is it possible to identify the object in $\mathcal{N}=4$ SYM for which both properties are manifest ?*

Yes! The expectation value of light-like Wilson loop in $\mathcal{N}=4$ SYM [Drummond-Henn-GK-Sokatchev]

$$W(C_4) = \frac{1}{N_c}\langle 0|\mathrm{Tr\,P}\exp\left(ig\oint_{C_4} dx^\mu A_\mu(x)\right)|0\rangle\,, \qquad C_4 =$$



✔ Gauge invariant functional of the integration contour C_4 in Minkowski space-time

✔ The contour is made out of 4 light-like segments $C_4 = \ell_1 \cup \ell_2 \cup \ell_3 \cup \ell_4$ joining the cusp points x_i^μ

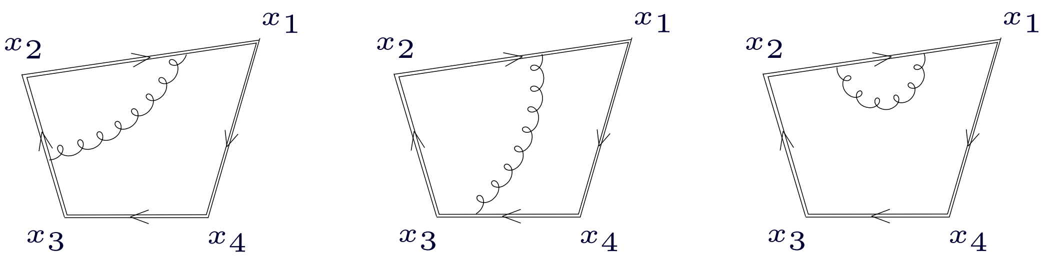
$$x_i^\mu - x_{i+1}^\mu = p_i^\mu = \text{on-shell gluon momenta}$$

✔ The contour C_4 has four light-like cusps $\mapsto$ $W(C_4)$ has UV divergencies

✔ Conformal symmetry of $\mathcal{N}=4$ SYM $\mapsto$ conformal invariance of $W(C_4)$ in dual coordinates x^μ

Gluon scattering amplitudes/Wilson loop duality I

The one-loop expression for the light-like Wilson loop (with $x_{jk}^2 = (x_j - x_k)^2$) [Drummond,GK,Sokatchev]

$\ln W(C_4) =$ 

$$= \frac{g^2}{4\pi^2} C_F \left\{ -\frac{1}{\epsilon_{\rm UV}{}^2} \left[(-x_{13}^2 \mu^2)^{\epsilon_{\rm UV}} + (-x_{24}^2 \mu^2)^{\epsilon_{\rm UV}} \right] + \frac{1}{2} \ln^2 \left(\frac{x_{13}^2}{x_{24}^2} \right) + \text{const} \right\} + O(g^4)$$

The one-loop expression for the gluon scattering amplitude

$$\ln \mathcal{A}_4(s,t) = \frac{g^2}{4\pi^2} C_F \left\{ -\frac{1}{\epsilon_{\rm IR}{}^2} \left[(-s/\mu_{\rm IR}^2)^{\epsilon_{\rm IR}} + (-t/\mu_{\rm IR}^2)^{\epsilon_{\rm IR}} \right] + \frac{1}{2} \ln^2 \left(\frac{s}{t} \right) + \text{const} \right\} + O(g^4)$$

✔ Identity the light-like segments with the on-shell gluon momenta $x_{i,i+1}^\mu \equiv x_i^\mu - x_{i+1}^\mu := p_i^\mu$:

$$x_{13}^2 \, \mu^2 := s/\mu_{\rm IR}^2 \, , \qquad x_{24}^2 \, \mu^2 := t/\mu_{\rm IR}^2 \, , \qquad x_{13}^2/x_{24}^2 := s/t$$

☞ UV divergencies of the light-like Wilson loop match IR divergences of the gluon amplitude

☞ the finite $\sim \ln^2(s/t)$ corrections coincide to one loop!

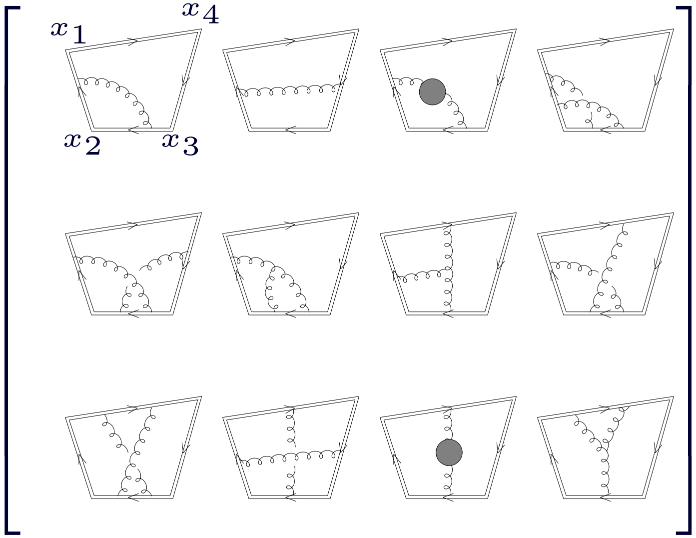
Gluon scattering amplitudes/Wilson loop duality II

Drummond-(Henn)-GK-Sokatchev proposal: *gluon amplitudes are dual to light-like Wilson loops*

$$\ln \mathcal{A}_4 = \ln W(C_4) + O(1/N_c^2, \epsilon_{\rm IR})\,.$$

✔ At strong coupling, the relation holds to leading order in $1/\sqrt{\lambda}$ [Alday,Maldacena]

✔ At weak coupling, the relation was verified to two loops [Drummond,Henn,GK,Sokatchev]

$\ln \mathcal{A}_4 = \ln W(C_4) =$  $= \frac{1}{4}\Gamma_{\rm cusp}(g)\ln^2(s/t) +$ Div

✔ Generalization to $n \geq 5$ gluon MHV amplitudes

$$\ln \mathcal{A}_n^{(\rm MHV)} = \ln W(C_n) + O(1/N_c^2)\,, \quad C_n = \text{light-like } n-\text{(poly)gon}$$

✗ At weak coupling, matches the BDS ansatz to one loop [Brandhuber,Heslop,Travaglini]

✗ The duality relation for $n = 5$ (pentagon) was verified to two loops [Drummond,Henn,GK,Sokatchev]

Conformal Ward identities for light-like Wilson loop

Main idea: *make use of conformal invariance of light-like Wilson loops in $\mathcal{N}=4$ SYM + duality relation to fix the finite part of $n-$gluon amplitudes*

- ✔ Conformal transformations map light-like polygon C_n into another light-like polygon C'_n
- ✔ Were the Wilson loop well-defined (=finite) in $D=4$ dimensions it would be conformal invariant

$$W(C_n) = W(C'_n)$$

- ✔ ... but $W(C_n)$ has cusp UV singularities $\mapsto$ dim.reg. breaks conformal invariance

$$W(C_n) = W(C'_n) \times \text{[cusp anomaly]}$$

- ✔ *All-loop* anomalous conformal Ward identities for the *finite part* of the Wilson loop

$$\ln W(C_n) = F_n^{(WL)} + \text{[UV divergencies]} + O(\epsilon)$$

Under special conformal transformations (boosts), to all orders, [Drummond,Henn,GK,Sokatchev]

$$\mathbb{K}^\mu \, F_n \equiv \sum_{i=1}^{n} \left[2x_i^\mu (x_i \cdot \partial_{x_i}) - x_i^2 \partial_{x_i}^\mu \right] F_n = \frac{1}{2} \Gamma_{\text{cusp}}(a) \sum_{i=1}^{n} x_{i,i+1}^\mu \ln\left(\frac{x_{i,i+2}^2}{x_{i-1,i+1}^2} \right)$$

The same relations also hold at strong coupling [Alday,Maldacena],[Komargodski]

Finite part of light-like Wilson loops

The consequences of the conformal Ward identity for the finite part of the Wilson loop W_n

- ✔ $n = 4, 5$ are special: there are no conformal invariants (too few distances due to $x_{i,i+1}^2 = 0$)
  $\Longrightarrow$ the Ward identity has a *unique all-loop solution* (up to an additive constant)

$$F_4 = \frac{1}{4}\Gamma_{\rm cusp}(a) \ln^2\left(\frac{x_{13}^2}{x_{24}^2}\right) + \text{ const },$$

$$F_5 = -\frac{1}{8}\Gamma_{\rm cusp}(a) \sum_{i=1}^{5} \ln\left(\frac{x_{i,i+2}^2}{x_{i,i+3}^2}\right) \ln\left(\frac{x_{i+1,i+3}^2}{x_{i+2,i+4}^2}\right) + \text{ const}$$

  Exactly the BDS ansatz for the 4- and 5-point MHV amplitudes!

- ✔ Starting from $n = 6$ there are conformal invariants in the form of cross-ratios

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}, \qquad u_2 = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2}, \qquad u_3 = \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2}$$

  General solution of the Ward identity contains *an arbitrary function* of the conformal cross-ratios.

- ✔ The BDS ansatz verifies the conformal Ward identity for *arbitrary* n but is it correct for $n \geq 6$?
  - ✗ does not match $n-$gon Wilson loop at strong coupling for $n \to \infty$ [Alday, Maldacena]
  - ✗ is not consistent with expected analytical properties of amplitudes [Bartels, Lipatov, Sabio Vera]

  If the BDS ansatz fails for $n = 6$, what is a missing function of $u_{1,2,3}$?

The hexagon lightlike Wilson loop

✔ Crucial test - go to six points at two loops where the answer is not determined by conformal symmetry. [Drummond,Henn,GK,Sokatchev'07]

$$F_6^{(\mathrm{WL})} = F_6^{(\mathrm{BDS})} + R^{(\mathrm{WL})}(u_1, u_2, u_3)$$

Result: The finite part of the Wilson loop has

✗ a non-trivial $R^{(\mathrm{WL})}(u_1, u_2, u_3) \neq 0$

✗ the correct behaviour in the imit where two momenta become collinear

✗ *Either the BDS conjecture or the amplitude/Wilson loop duality fails from six points and beyond*

✔ Two-loop calculation of the 6-gluon amplitude for a few kinematical points $R^{(\mathrm{MHV})} \neq 0$ [Bern,Dixon,Kosower,Roiban,Spradlin,Vergu,Volovich'08]

✔ Numerical comparison between the discrepancy functions $R^{(\mathrm{WL})}$ and $R^{(\mathrm{MHV})}$ shows that they coincide with an accuracy $< 10^{-4}$

$$F_6^{(\mathrm{WL})} = F_6^{(\mathrm{MHV})}$$

✌ *The Wilson loop/gluon scattering amplitude duality holds at $n = 6$ to two loops!*

✌ *Strong evidence that the duality relation also holds for arbitrary n to all loops!*

From $\mathcal{N}=4$ SYM to QCD

Finite part of 4-gluon amplitude in QCD at two loops ($x=-\frac{t}{s}$, $y=-\frac{u}{s}$, $z=-\frac{u}{t}$, $X=\ln x$, $Y=\ln y$, $S=\ln z$)

[Glover,Oleari,Tejeda-Yeomans'01]

$$
\begin{aligned}
\mathcal{M}_4^{(\mathrm{QCD})} = \Big\{ \Big(& 48\,\mathrm{Li}_4(x) - 48\,\mathrm{Li}_4(y) - 128\,\mathrm{Li}_4(z) + 40\,\mathrm{Li}_3(x)\,X - 64\,\mathrm{Li}_3(x)\,Y - \tfrac{98}{3}\,\mathrm{Li}_3(x) + 64\,\mathrm{Li}_3(y)\,X - 40\,\mathrm{Li}_3(y)\,Y \\
& + 18\,\mathrm{Li}_3(y) + \tfrac{98}{3}\,\mathrm{Li}_2(x)\,X - \tfrac{16}{3}\,\mathrm{Li}_2(x)\,\pi^2 - 18\,\mathrm{Li}_2(y)\,Y - \tfrac{37}{6}\,X^4 + 28\,X^3\,Y - \tfrac{23}{3}\,X^3 - 16\,X^2\,Y^2 + \tfrac{49}{3}\,X^2\,Y - \tfrac{35}{3}\,X^2\,\pi^2 - \tfrac{38}{3}\,X^2 \\
& - \tfrac{22}{3}\,S\,X^2 - \tfrac{20}{3}\,X\,Y^3 - 9\,X\,Y^2 + 8\,X\,Y\,\pi^2 + 10\,X\,Y - \tfrac{31}{12}\,X\,\pi^2 - 22\,\zeta_3\,X + \tfrac{22}{3}\,S\,X + \tfrac{37}{27}\,X + \tfrac{11}{6}\,Y^4 - \tfrac{41}{9}\,Y^3 - \tfrac{11}{3}\,Y^2\,\pi^2 \\
& - \tfrac{22}{3}\,S\,Y^2 + \tfrac{266}{9}\,Y^2 - \tfrac{35}{12}\,Y\,\pi^2 + \tfrac{418}{9}\,S\,Y + \tfrac{257}{9}\,Y + 18\,\zeta_3\,Y - \tfrac{31}{30}\,\pi^4 - \tfrac{11}{9}\,S\,\pi^2 + \tfrac{31}{9}\,\pi^2 + \tfrac{242}{9}\,S^2 + \tfrac{418}{9}\,\zeta_3 + \tfrac{2156}{27}\,S \\
& - \tfrac{11093}{81} - 8\,S\,\zeta_3 \Big) \frac{t^2}{s^2} + \Big(-256\,\mathrm{Li}_4(x) - 96\,\mathrm{Li}_4(y) + 96\,\mathrm{Li}_4(z) + 80\,\mathrm{Li}_3(x)\,X + 48\,\mathrm{Li}_3(x)\,Y - \tfrac{64}{3}\,\mathrm{Li}_3(x) - 48\,\mathrm{Li}_3(y)\,X \\
& + 96\,\mathrm{Li}_3(y)\,Y - \tfrac{304}{3}\,\mathrm{Li}_3(y) + \tfrac{64}{3}\,\mathrm{Li}_2(x)\,X - \tfrac{32}{3}\,\mathrm{Li}_2(x)\,\pi^2 + \tfrac{304}{3}\,\mathrm{Li}_2(y)\,Y + \tfrac{26}{3}\,X^4 - \tfrac{64}{3}\,X^3\,Y - \tfrac{64}{3}\,X^3 + 20\,X^2\,Y^2 \\
& + \tfrac{136}{3}\,X^2\,Y + 24\,X^2\,\pi^2 + 76\,X^2 - \tfrac{88}{3}\,S\,X^2 + \tfrac{8}{3}\,X\,Y^3 + \tfrac{104}{3}\,X\,Y^2 - \tfrac{16}{3}\,X\,Y\,\pi^2 + \tfrac{176}{3}\,S\,X\,Y - \tfrac{136}{3}\,X\,Y - \tfrac{50}{3}\,X\,\pi^2 \\
& - 48\,\zeta_3\,X + \tfrac{2350}{27}\,X + \tfrac{440}{3}\,S\,X + 4\,Y^4 - \tfrac{176}{9}\,Y^3 + \tfrac{4}{3}\,Y^2\,\pi^2 - \tfrac{176}{3}\,S\,Y^2 - \tfrac{494}{9}\,Y\,\pi^2 + \tfrac{5392}{27}\,Y - 64\,\zeta_3\,Y + \tfrac{496}{45}\,\pi^4 \\
& - \tfrac{308}{9}\,S\,\pi^2 + \tfrac{200}{9}\,\pi^2 + \tfrac{968}{9}\,S^2 + \tfrac{8624}{27}\,S - \tfrac{44372}{81} + \tfrac{1864}{9}\,\zeta_3 - 32\,S\,\zeta_3 \Big) \frac{t}{u} + \Big(\tfrac{88}{3}\,\mathrm{Li}_3(x) - \tfrac{88}{3}\,\mathrm{Li}_2(x)\,X + 2\,X^4 - 8\,X^3\,Y \\
& - \tfrac{220}{9}\,X^3 + 12\,X^2\,Y^2 + \tfrac{88}{3}\,X^2\,Y + \tfrac{8}{3}\,X^2\,\pi^2 - \tfrac{88}{3}\,S\,X^2 + \tfrac{304}{9}\,X^2 - 8\,X\,Y^3 - \tfrac{16}{3}\,X\,Y\,\pi^2 + \tfrac{176}{3}\,S\,X\,Y - \tfrac{77}{3}\,X\,\pi^2 \\
& + \tfrac{1616}{27}\,X + \tfrac{968}{9}\,S\,X - 8\,\zeta_3\,X + 4\,Y^4 - \tfrac{176}{9}\,Y^3 - \tfrac{20}{3}\,Y^2\,\pi^2 - \tfrac{176}{3}\,S\,Y^2 - \tfrac{638}{9}\,Y\,\pi^2 - 16\,\zeta_3\,Y + \tfrac{5392}{27}\,Y - \tfrac{4}{15}\,\pi^4 - \tfrac{308}{9}\,S\,\pi^2 \\
& - 20\,\pi^2 - 32\,S\,\zeta_3 + \tfrac{1408}{9}\,\zeta_3 + \tfrac{968}{9}\,S^2 - \tfrac{44372}{81} + \tfrac{8624}{27}\,S \Big) \frac{t^2}{u^2} + \Big(\tfrac{44}{3}\,\mathrm{Li}_3(x) - \tfrac{44}{3}\,\mathrm{Li}_2(x)\,X - X^4 + \tfrac{110}{9}\,X^3 - \tfrac{22}{3}\,X^2\,Y \\
& + \tfrac{14}{3}\,X^2\,\pi^2 + \tfrac{44}{3}\,S\,X^2 - \tfrac{152}{9}\,X^2 - 10\,X\,Y + \tfrac{11}{2}\,X\,\pi^2 + 4\,\zeta_3\,X - \tfrac{484}{9}\,S\,X - \tfrac{808}{27}\,X + \tfrac{7}{30}\,\pi^4 - \tfrac{31}{9}\,\pi^2 \\
& + \tfrac{11}{9}\,S\,\pi^2 - \tfrac{418}{9}\,\zeta_3 - \tfrac{242}{9}\,S^2 - \tfrac{2156}{27}\,S + 8\,S\,\zeta_3 + \tfrac{11093}{81} \Big) \frac{u\,t}{s^2} + \Big(-176\,\mathrm{Li}_4(x) + 88\,\mathrm{Li}_3(x)\,X - 168\,\mathrm{Li}_3(x)\,Y - \ldots
\end{aligned}
$$

No relation to rectangular Wilson loop ... but let us examine the Regge limit $s \gg -t$

Four-gluon amplitude/Wilson loop duality in QCD

- ✔ Planar four-gluon QCD scattering amplitude in the Regge limit $s \gg -t$ [Schnitzer'76],[Fadin,Kuraev,Lipatov'76]

$$\mathcal{M}_4^{(\mathrm{QCD})}(s,t) \sim (s/(-t))^{\omega_R(-t)} + \ldots$$

  The Regge trajectory $\omega_R(-t)$ is known to two loops [Fadin,Fiore,Kotsky'96]

- ✔ The all-loop gluon Regge trajectory in QCD [GK'96]

$$\omega_R(-t) = \frac{1}{2}\int_{(-t)}^{\mu_{\mathrm{IR}}^2} \frac{dk_\perp^2}{k_\perp^2}\Gamma_{\mathrm{cusp}}(a(k_\perp^2)) + \Gamma_R(a(-t)) + [\text{poles in } 1/\epsilon_{\mathrm{IR}}]\,,$$

- ✔ Rectangular Wilson loop in QCD in the Regge limit $|x_{13}^2| \gg |x_{24}^2|$

$$W^{(\mathrm{QCD})}(C_4) \sim \left(x_{13}^2/(-x_{24}^2)\right)^{\omega_{\mathrm{W}}(-x_{24}^2)} + \ldots$$

- ✔ The all-loop Wilson loop 'trajectory' in QCD

$$\omega_{\mathrm{W}}^{(\mathrm{QCD})}(-t) = \frac{1}{2}\int_{(-t)}^{\mu_{\mathrm{UV}}^2} \frac{dk_\perp^2}{k_\perp^2}\Gamma_{\mathrm{cusp}}(a(k_\perp^2)) + \Gamma_{\mathrm{W}}(a(-t)) + [\text{poles in } 1/\epsilon_{\mathrm{UV}}]\,,$$

- ✔ *The scattering amplitude/Wilson loop duality relation holds in QCD in the Regge limit only* [GK'96]

$$\ln \mathcal{M}_4^{(\mathrm{QCD})}(s,t) = \ln W^{(\mathrm{QCD})}(C_4) + O(t/s)$$

  while in $\mathcal{N}=4$ SYM it is exact for arbitrary t/s!

From MHV amplitudes to MHV superamplitude

- On-shell helicity states in $\mathcal{N}=4$ SYM:

$$G^{\pm} \text{ (gluons } h=\pm 1), \qquad \Gamma_A,\ \bar{\Gamma}^A \text{ (gluinos } h=\tfrac{1}{2}), \qquad S_{AB} \text{ (scalars } h=0)$$

- Can be combined into a single on-shell superstate [Mandelstam],[Brink et el]

$$\Phi(p,\eta) = G^{+}(p) + \eta^A \Gamma_A(p) + \frac{1}{2}\eta^A\eta^B S_{AB}(p) + \frac{1}{3!}\eta^A\eta^B\eta^C \epsilon_{ABCD}\bar{\Gamma}^D(p) + \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D\epsilon_{ABCD}G^{-}(p)$$

- Combine all MHV amplitudes into a single MHV superamplitude [Nair]

$$\mathcal{A}_n^{\mathrm{MHV}} = (\eta_1)^4(\eta_2)^4 \times A\left(G_1^- G_2^- G_3^+ \dots G_n^+\right) + (\eta_1)^4(\eta_2)^2(\eta_3)^2 \times A\left(G_1^- \bar{S}_2\, S_3 \dots G_n^+\right) + \dots$$

- Spinor helicity formalism: commuting spinors λ^α (helicity -1/2), $\tilde{\lambda}^{\dot\alpha}$ (helicity 1/2) [Xu,Zhang,Chang'87]

$$p_i^2 = 0 \quad \Leftrightarrow \quad p_i^{\alpha\dot\alpha} \equiv p_i^\mu (\sigma_\mu)^{\alpha\dot\alpha} = \lambda_i^\alpha\, \tilde{\lambda}_i^{\dot\alpha}$$

- On-shell $\mathcal{N}=4$ supersymmetry:

$$q_\alpha^A = \sum_i \lambda_{i,\alpha}\eta_i^A\,, \qquad \bar{q}_{A\,\dot\alpha} = \sum_i \tilde{\lambda}_{i,\dot\alpha}\frac{\partial}{\partial\eta_i^A} \qquad \Longrightarrow \qquad q_\alpha^A\,\mathcal{A}_n^{\mathrm{MHV}} = \bar{q}_{A\,\dot\alpha}\mathcal{A}_n^{\mathrm{MHV}} = 0$$

All-loop MHV superamplitude

- ✔ All MHV amplitudes are combined into a single superamplitude (spinor notations $\langle ij\rangle = \lambda_i^\alpha \lambda_{j\,\alpha}$)

$$\mathcal{A}_n^{\rm MHV}(p_1,\eta_1;\ldots;p_n,\eta_n) = i\,\frac{\delta^{(4)}\left(\sum_{i=1}^n p_i\right)\,\delta^{(8)}\left(\sum_{i=1}^n \lambda_i^\alpha \eta_i^A\right)}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}\,M_n^{(\rm MHV)}\,,$$

  - ✗ Perturbative corrections to all MHV amplitudes are factorized into a universal factor $M_n^{(\rm MHV)}$
  - ✗ The all-loop MHV amplitudes appear as coefficients in the expansion of $\mathcal{A}_n^{\rm MHV}$ in powers of η's

$$\mathcal{A}_n^{\rm MHV} = \delta^{(4)}\Big(\sum_{i=1}^n p_i\Big)\sum_{1\le j<k\le n}(\eta_j)^4(\eta_k)^4 A_n^{(\rm MHV)}(1^+\ldots j^-\ldots k^-\ldots n^+) + \ldots\,, \qquad (1)$$

  - ✗ The function $M_n^{(\rm MHV)}$ is dual to light-like $n-$gon Wilson loop

$$\ln M_n^{(\rm MHV)} = \ln W_n + O(\epsilon, 1/N^2)$$

- ✔ The MHV superamplitude possesses a much bigger, dual superconformal symmetry which acts on the dual coordinates x_i^μ and their superpartners $\theta_{i\,\alpha}^A$ [Drummond, Henn, GK, Sokatchev]

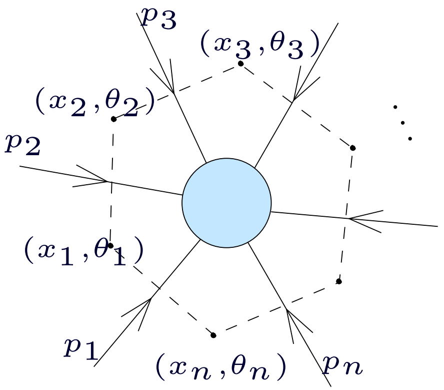
$$p_i^\mu = x_i^\mu - x_{i+1}^\mu\,, \qquad \lambda_i^\alpha\,\eta_i = \theta_i^\alpha - \theta_{i+1}^\alpha$$

Dual $\mathcal{N}=4$ superconformal symmetry I

✔ Tree-level MHV superamplitude (with $\langle ij\rangle = \lambda_i^{\alpha}\lambda_{j\,a}$)

$$\mathcal{A}_n^{\mathrm{MHV;tree}} = i\,\frac{\delta^{(4)}\left(\sum_{i=1}^n p_i\right)\,\delta^{(8)}\left(\sum_{i=1}^n \lambda_i^{\alpha}\eta_i^A\right)}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}$$

✔ Chiral dual superspace $(x_{\alpha\dot\alpha}\,,\,\theta_\alpha^A\,,\,\lambda_\alpha)$:



✗ $p = \sum_{i=1}^n p_i = 0 \;\rightarrow\; p_i = x_i - x_{i+1}\,,\quad x_{n+1} = x_1$

✗ $q = \sum_{i=1}^n \lambda_i\eta_i = 0 \;\rightarrow\; \lambda_{i\,\alpha}\,\eta_i^A = (\theta_i - \theta_{i+1})_\alpha^A\,,\quad \theta_{n+1} = \theta_1$

✔ The MHV superamplitude in the dual superspace

$$\mathcal{A}_n^{\mathrm{MHV;tree}} = i(2\pi)^4\,\frac{\delta^{(4)}\left(x_1 - x_{n+1}\right)\,\delta^{(8)}\left(\theta_1 - \theta_{n+1}\right)}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}$$

✔ $\mathcal{N}=4$ supersymmetry in the dual superspace:

$$Q_{A\,\alpha} = \sum_{i=1}^n \frac{\partial}{\partial\theta_i^{A\,\alpha}}, \qquad \bar{Q}^A_{\dot\alpha} = \sum_{i=1}^n \theta_i^{A\,\alpha}\frac{\partial}{\partial x_i^{\dot\alpha\alpha}}, \qquad P_{\alpha\dot\alpha} = \sum_{i=1}^n \frac{\partial}{\partial x_i^{\dot\alpha\alpha}}$$

Dual $\mathcal{N} = 4$ superconformal symmetry II

- Super-Poincaré $+$ inversion $=$ conformal supersymmetry:
  - Inversions in the dual superspace

$$I[\lambda_i^{\alpha}] = (x_i^{-1})^{\dot{\alpha}\beta}\lambda_{i\beta}\,, \qquad I[\theta_i^{\alpha\,A}] = (x_i^{-1})^{\dot{\alpha}\beta}\theta_i^{\beta\,A}$$

  - Neighbouring contractions are dual conformal covariant

$$I[\langle i\, i+1\rangle] = (x_i^2)^{-1}\langle i\, i+1\rangle$$

  - Impose cyclicity, $x_{n+1} = x_1$, $\theta_{n+1} = \theta_1$, through delta functions. Then, only in $\mathcal{N} = 4$,

$$I[\delta^{(4)}(x_1 - x_{n+1})] = x_1^8\, \delta^{(4)}(x_1 - x_{n+1}) \qquad I[\delta^{(8)}(\theta_1 - \theta_{n+1})] = x_1^{-8}\, \delta^{(8)}(\theta_1 - \theta_{n+1})$$

- MHV superamplitude in dual superspace

$$\mathcal{A}_n^{\mathrm{MHV}}(x,\theta,\lambda) = \frac{\delta^{(4)}(x_1 - x_{n+1})\,\delta^{(8)}(\theta_1 - \theta_{n+1})}{\langle 1\,2\rangle\langle 2\,3\rangle \ldots \langle n\,1\rangle}\; M_n^{\mathrm{loops}}(x_{ij})$$

  - Tree – manifestly dual superconformal covariant.
  - Loops – IR divergent factor $M_n^{\mathrm{loops}}(x_{ij})$ satisfies anomalous dual conformal Ward identity

- Dual Poincaré supersymmetry $\bar{Q} = \sum_i \theta_i \partial/\partial x_i$ is a symmetry of the tree, not of the loops !?! Needs better understanding !

- Our conjecture: dual superconformal covariance is a property of all tree-level superamplitudes (NMHV, N^2MHV, ...) in $\mathcal{N} = 4$ SYM theory

Conclusions and recent developments

- ✔ MHV amplitudes in $\mathcal{N}=4$ theory
  - ✗ possess the dual conformal symmetry both at weak and at strong coupling
  - ✗ Dual to light-like Wilson loops

  ... but what about NMHV, NNMHV, *etc.* amplitudes?

- ✔ This symmetry is a part of much bigger dual superconformal symmetry of all planar superamplitudes in $\mathcal{N}=4$ SYM [Drummond,Henn,GK,Sokatchev]
  - ✗ Relates various particle amplitudes with different helicity configurations (MHV, NMHV,...)
  - ✗ Imposes non-trivial constraints on the loop corrections

- ✔ Dual superconformal symmetry is now explained better through the AdS/CFT correspondence by a combined bosonic [Kallosh,Tseytlin] and fermionic T duality symmetry [Berkovits,Maldacena], [Beisert,Ricci,Tseytlin,Wolf]

- ✔ Dual symmetry is also present in QCD but in the Regge limit only ... yet another glimpse of QCD/string duality?!

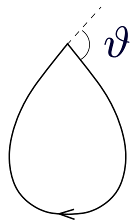
- ✔ What is the generalisation of the Wilson loop/amplitude duality beyond MHV?

Back-up slides

What is the cusp anomalous dimension

- ✔ Cusp anomaly is a very 'unfortunate' feature of Wilson loops evaluated over an *Euclidean* closed contour with a cusp – generates the anomalous dimension [Polyakov'80]

$$\langle \mathrm{tr}\, \mathsf{P} \exp\left(i \oint_C dx \cdot A(x) \right) \rangle \sim (\Lambda_{\rm UV})^{\Gamma_{\sf cusp}(g,\vartheta)} \,, \qquad C =$$



- ✔ A very 'fortunate' property of Wilson loop – the cusp anomaly controls the *infrared* asymptotics of scattering amplitudes in gauge theories [GK, Radyushkin'86]
  - ✗ The integration contour C is defined by the particle momenta
  - ✗ The cusp angle ϑ is related to the scattering angles in *Minkowski* space-time, $|\vartheta| \gg 1$

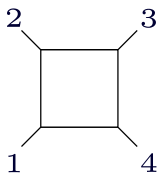
$$\Gamma_{\rm cusp}(g,\vartheta) = \vartheta\, \Gamma_{\rm cusp}(g) + O(\vartheta^0)\,,$$

- ✔ *The cusp anomalous dimension* $\Gamma_{\rm cusp}(g)$ is an ubiquitous observable in gauge theories: [GK'89]
  - ✗ Logarithmic scaling of anomalous dimensions of high-spin Wilson operators;
  - ✗ IR singularities of on-shell gluon scattering amplitudes;
  - ✗ Gluon Regge trajectory;
  - ✗ Sudakov asymptotics of elastic form factors;
  - ✗ ...

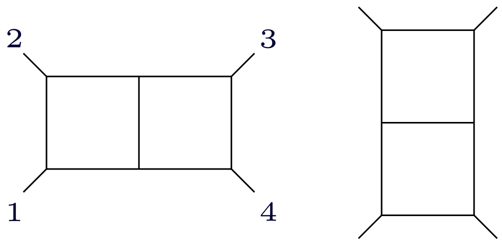
Four-gluon planar amplitude at weak coupling

Weak coupling corrections to $A_4/A_4^{(0)}$ can be expressed in terms of scalar integrals:

- ✔ One loop: [Green,Schwarz,Brink'82]

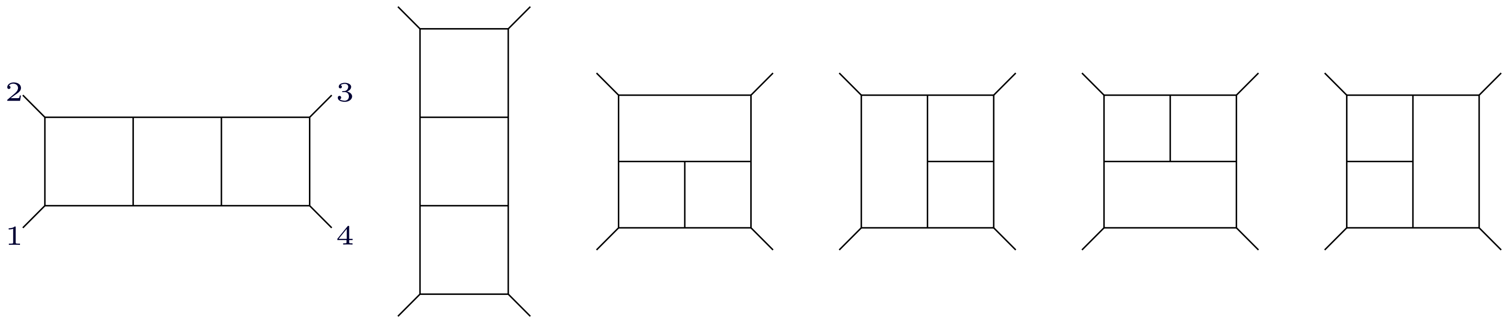


- ✔ Two loops: [Bern,Rozowsky,Yan'97]



all-loop iteration structure conjectured [Anastasiou,Bern,Dixon,Kosower'03]

- ✔ Three loops: [Bern,Dixon,Smirnov'05]



iteration structure confirmed!

- ✔ Four loops: scalar integrals of 8 different topologies are identified [Bern,Czakov,Dixon,Kosower,Smirnov'06]

Light-like Wilson loops

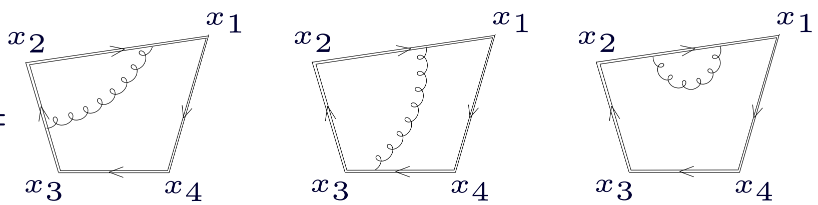
To lowest order in the coupling constant,

$$W(C_4) = 1 + \frac{1}{2}(ig)^2 C_F \sum_{1\le j,\, k\le 4} \int_{\ell_j} dx^\mu \int_{\ell_k} dy^\nu \, G_{\mu\nu}(x-y) + O(g^4)\,, \tag{2}$$

- ✔ The gluon propagator in the coordinate representation (the Feynman gauge + dimensional regularization, $D = 4 - 2\epsilon$)

$$G_{\mu\nu}(x) = -g_{\mu\nu} \frac{\Gamma(1-\epsilon)}{4\pi^2} (-x^2 + i0)^{-1+\epsilon} (\mu^2 \pi)^\epsilon\,.$$

- ✔ Feynman diagram representation

$\ln W(C_4) =$ 

- ✔ The light-like Wilson loop is IR finite but has UV divergences due to cusps on the integration contour C_4

$$\ln W(C_4) = \frac{g^2}{4\pi^2} C_F \left\{ -\frac{1}{2\epsilon^2} \sum_{i=1}^{4} \left(-x_{i-1,i+1}^2\, \mu^2\right)^\epsilon + O(\epsilon^0) \right\} + O(g^4)\,.$$