Conductivity and Diffusion in Holographic Flavor Systems

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Overview

- Introdution and motivation
- Einstein relation and gravitational dual
- Holographic Dp/Dq flavor setup
- Diffusion, conductivity and susceptibility
- Conclusions

Invitation

- There is a established relation between *black hole geometry* and *equilibrium thermodynamics*.

$$dM = \frac{\kappa}{8\pi G} dA + \Omega_H dJ \quad \longleftrightarrow \quad d\epsilon = T ds + \mu d\rho$$

Euclidean action $\longleftrightarrow$ Free energy

- Does the horizon dynamics capture *near - equilibrium thermodynamics* as well ?

Quasinormal modes Membrane paradigm	$\longleftrightarrow$	Transport coefficients Fluctuation dissipation th. ....

- AdS-CFT correspondence gives a guide:

1- helps pose well defined questions from QFT onto gravity and viceversa
2- strong consistency checks both ways (Onsager's relations, relativistic hydro.)

Einstein relation

Consider a particle moving in a medium $$m\,\dot{v}(t) = -\gamma\, v(t) + F(t)$$

For a constant force we can study conduction

$$F(t) = F_0 \quad \Rightarrow \quad v \xrightarrow{t\to\infty} \mu F_0 \qquad\qquad \mu = \frac{1}{\gamma}$$

For a random force we can study diffusion

$$\langle F(t)\rangle = 0 \quad \Rightarrow \quad \langle x^2\rangle(t) \xrightarrow{t\to\infty} 2(k_B T \mu)\, t$$

Einstein relation $$D = \mu \left(\frac{1}{k_B T}\right)^{-1}$$

more generally $$\boxed{D = \frac{\sigma}{\chi}}$$ where the equilibrium susceptibility $$\chi = \left.\frac{\partial \rho}{\partial \mu}\right|_{T,\mu=0}$$

Einstein relation

An argument for the Einstein relation comes from comparing

$$\mathbf{J}_\mu = D\nabla\rho = D\frac{\partial\rho}{\partial\mu}\nabla\mu = D\chi\nabla\mu$$

and

$$\mathbf{J}_\Phi = \sigma\mathbf{E} = \sigma\nabla\Phi$$

in equilibrium $\mathbf{J}_\mu = \mathbf{J}_\phi \;\Leftrightarrow\; \mu = \Phi$ or $D = \dfrac{\sigma}{\chi}$

Questions

1. How general is this relation: no-quasiparticles, strong coupling, etc.
2. Is this relation still valid at finite chemical potential $\mu = \mu_0$ with the prescription for
$$\chi = \left.\frac{\partial\rho}{\partial\mu}\right|_{T,\mu=\mu_0}$$
3. Can it be naturally mapped onto the geometry of black holes?
4. Same for other generalized Einstein relations
$$D_T = \sigma_T/C_p \qquad\qquad D_p = \eta/\chi_p$$

Einstein relation and membrane paradigm

The static geometry of black holes contains information about thermodynamics

$$ds^2 = g_{00}(r)dt^2 + g_{rr}(r)dr^2 + g_{ii}(r)d\vec{x}_d^2$$

$$\boxed{T_H = -\frac{1}{4\pi}\frac{g'_{00}}{\sqrt{g_{00}g_{rr}}}\Big|_{r_H}}$$

It also contains information about transport coefficients close to the horizon (membrane paradigm)

$$S \sim \int d^{d+1}x\,\sqrt{-g}\left(R - \frac{1}{4g_{d+1}^2}F^{\mu\nu}F_{\mu\nu} + \ldots\right) - \int_{r_H+\epsilon} d^dx\sqrt{-\gamma}\left(\frac{1}{g_{d+1}^2}\sqrt{g_{rr}}F_{r\mu}\right)A^\mu$$

$$J_\mu = -\frac{\sqrt{g_{rr}}}{g_{d+1}^2}F_{r\mu} \qquad \boxed{J_i = -D\partial_i J_0}$$

$$\boxed{D = \frac{\sqrt{-g}}{g_{ii}\sqrt{-g_{00}g_{rr}}}\Big|_{r_H} \times \int_{r_H}^{\infty}\frac{-g_{00}(r)g_{rr}(r)}{\sqrt{-g(r)}}}$$

P. Kovtun D.T. Son & A. O. Starinets (2003)

$$\overset{D3/D7}{\hookrightarrow} = \sigma\,\chi^{-1}$$

R. Myers, A. O. Starinets & R. M. Thomson (2007)

The relation with AdS/CFT remained unclear until recently

A.O. Starinets (2006)

N. Iqbal & H. Liu (2008)

Diffusion and conductivity from correlators

Linear response theory allows to compute transport coefficients in terms of retarded Green's function. For a vector current J^μ

$$G_R^{\mu,\nu}(\omega,\mathbf{q}) = -i\int \frac{d^D x}{(2\pi)^D} e^{ikx}\theta(x^0-y^0)\langle[J^\mu(x),J^\nu(0)]\rangle$$

In a thermal vacuum rotational symmetry leaves a basis of two scalar functions

$$G^R_{\mu\nu}(\omega,\mathbf{q}) = P^\perp_{\mu\nu}\,\Pi^\perp(\omega,\mathbf{q}) + P^{||}_{\mu\nu}\,\Pi^{||}(\omega,\mathbf{q})$$

for example for $k^\mu = (\omega,0,0,q)$

$$G_{tt} = \tfrac{q^2}{\omega^2-q^2}\Pi^{||}(\omega,q)\ ;\quad G_{zz} = \tfrac{\omega^2}{\omega^2-q^2}\Pi^{||}(\omega,q)$$

$$G_{tz} = \tfrac{\omega q}{\omega^2-q^2}\Pi^{||}(\omega,q)\ ;\quad G_{xx} = G_{yy} = \Pi^\perp(\omega,q)$$

Diffusion and conductivity from correlators

The DC conductivity measures the response to an external field (Ohm's law)

$$\mathbf{J} = \sigma\,\mathbf{E}$$

Microscopically it can be obtained from $\Pi_\perp$

$$\sigma \equiv \lim_{\omega\to 0} \sigma(\omega) = \lim_{\omega\to 0} \frac{\mathrm{Im}\,\Pi_\perp(\omega, q=0)}{\omega}$$

The diffusion constant measures the current generated in response to a density gradient

$$\mathbf{J} = -D\nabla\rho$$

In the "hydro" limit, the diffusion coefficient shows up as a pole in $\Pi_{||}$

$$\begin{array}{ccc} \omega & \to & 0 \\ \mathbf{q} & \to & 0 \end{array}$$

$$\Pi_{||} \sim \frac{C}{i\omega - Dq^2} + \ldots$$

The case of the R charged black hole

The simplest example with chemical potential is the R-charged black hole. The STU black hole has $U(1)^3$

For perturbations like $\sim e^{-i(\omega t + qz)}$ we expect them to group up as follows

$$\text{Vector} \quad \Rightarrow \quad A_x\,,\, h_{tx}\,,\, h_{z,x} \quad \text{and}\ (x \leftrightarrow y) \quad \longrightarrow \quad \Pi_\perp$$

$$\text{Scalar} \quad \Rightarrow \quad A_t\,,\, A_z\,,\, h_{tt}\,,\, h_{tz}\,,\, h_{xx} = h_{yy}\,,\, h_{zz}\,, \quad \longrightarrow \quad \Pi_{||}$$

The charge (κ, κ, κ) case has been analyzed in

Ge, Matsuo, Shu, Sin &Tsukioka (2008)

Matsuo, Sin , Takeuchi, Tsukioka & Yoo (2009)

$$\sigma \equiv \lim_{\omega \to 0} \frac{\operatorname{Im} \Pi_\perp(\omega, \mathbf{q}=0)}{\omega} = \frac{N_c^2 T_0}{16\pi} \frac{(\kappa-2)^2}{(\kappa+1)^2}$$

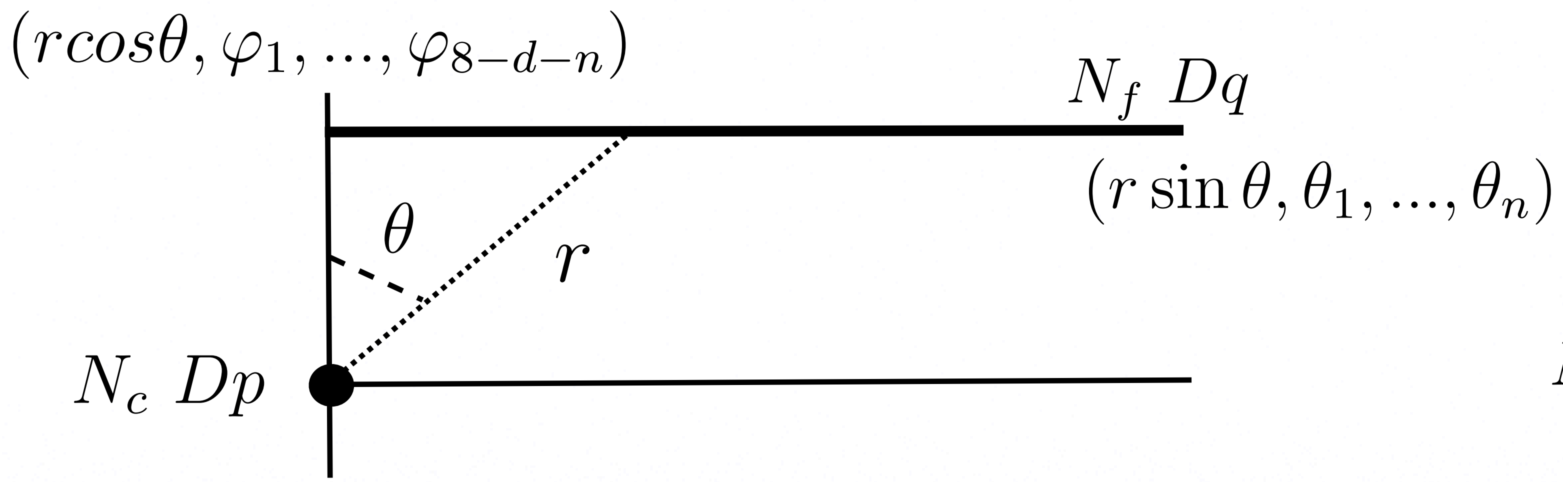
$$D = \frac{1}{4\pi T_0} \frac{\kappa+2}{\kappa+1}$$

$$\chi = \frac{\sigma}{D} = \frac{N_c^2 T_0^2}{4} \frac{(\kappa-2)^2}{(\kappa+1)(\kappa+2)}$$

$$\chi_{Th.} = \left.\frac{\partial \langle n_q \rangle}{\partial \mu}\right|_{T,\mu_0} = \left.\frac{\partial \Omega(T,\mu)}{\partial \mu^2}\right|_{T,\mu_0}$$

$$= \frac{N_c^2 T_0^2}{4} \frac{5\kappa+2}{\kappa+2}$$

Holographic Dp/Dq flavor system: background



$N_f << N_c$

Parametrize the Dq brane profile through $\cos\theta(r) = \psi(r)$

The $q+1 = 1+d+1+n$ dimensional induced metric adapted to intersection is given by

$$ds^2 = \tilde{g}_{00}(r)dt^2 + \tilde{g}_{ii}(r)d\vec{x}_d^2 + \tilde{g}_{rr}(r)dr^2 + \tilde{g}_{\theta\theta}(r)d\Omega_n^2 + \qquad \tilde{g}_{ab} = \tilde{g}_{ab}(r,\psi(r))$$

The dynamics is governed by

$$S_{DBI} = -N_f T_{D_q}\int d^{d+2+n}x\,\sqrt{-\gamma} \qquad \gamma_{ab} = \tilde{g}_{ab} + (2\pi\alpha')F_{ab}$$

This allows for several gauge field configurations. For a chemical potential $\gamma_{0r} = 2\pi\alpha' A_0'(r)$

The degrees of freedom (ψ, A_0) obey equations of motion

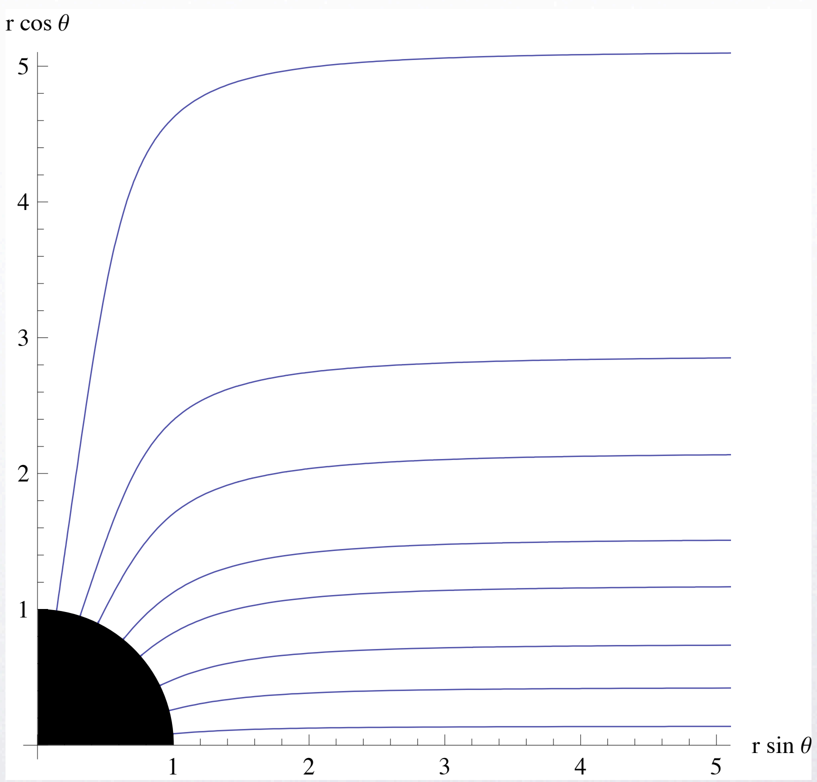
$$\psi \quad \rightarrow \quad \left(e^{-\phi}\sqrt{\gamma}\psi'\gamma^{rr}\gamma_{\psi\psi}\right)' - \tfrac{1}{2}e^{-\phi}\sqrt{\gamma}\left(\psi'^2\gamma^{rr}\gamma_{\psi\psi,\psi} + n\gamma^{\theta\theta}\gamma_{\theta\theta,\psi}\right) = 0$$

$$A_0 \quad \rightarrow \quad \left(e^{-\phi}\sqrt{\gamma}\gamma^{0r}\right)' = 0$$

which admit a constant charge density $n_q = \frac{\mathcal{N}}{2\pi\alpha'}\left(e^{-\phi}\sqrt{-\gamma}\gamma^{0r}\right) \equiv \frac{\delta S_{DBI}}{\delta A_0'(r)}$ $\mathcal{N} = N_f T_{D_q}(2\pi\alpha')^2\Omega_n$

$$\psi \sim \frac{m}{r^{2/(7-p)}} + \frac{c}{r^{2n/(7-p)}} + \ldots \qquad \begin{cases} m \sim M_q/T \\ c \sim \langle\bar{\psi}\psi\rangle \end{cases}$$

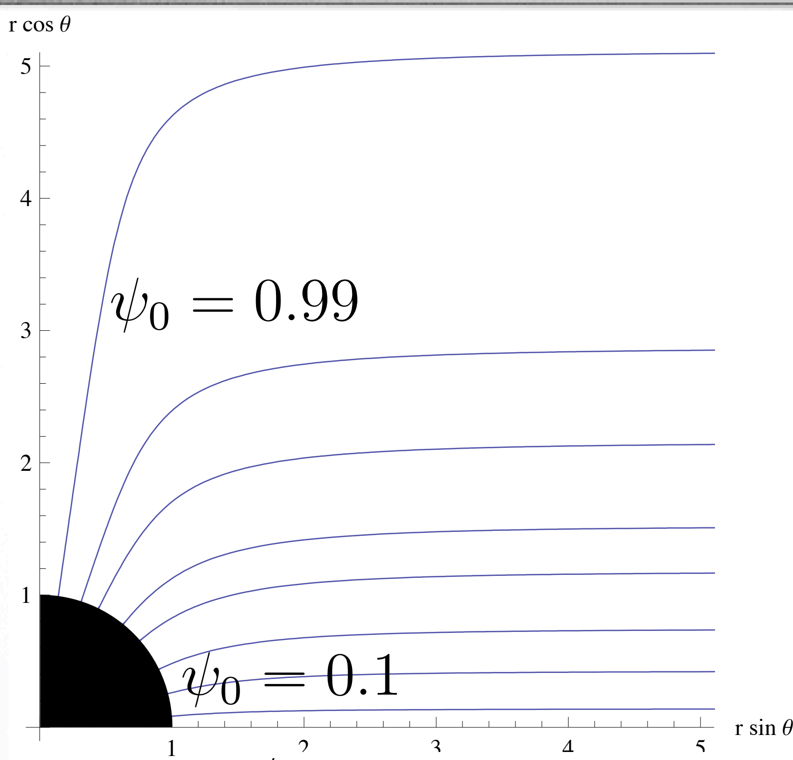
$$A_0 \sim \mu + \frac{n_q}{r^\beta} + \ldots.$$

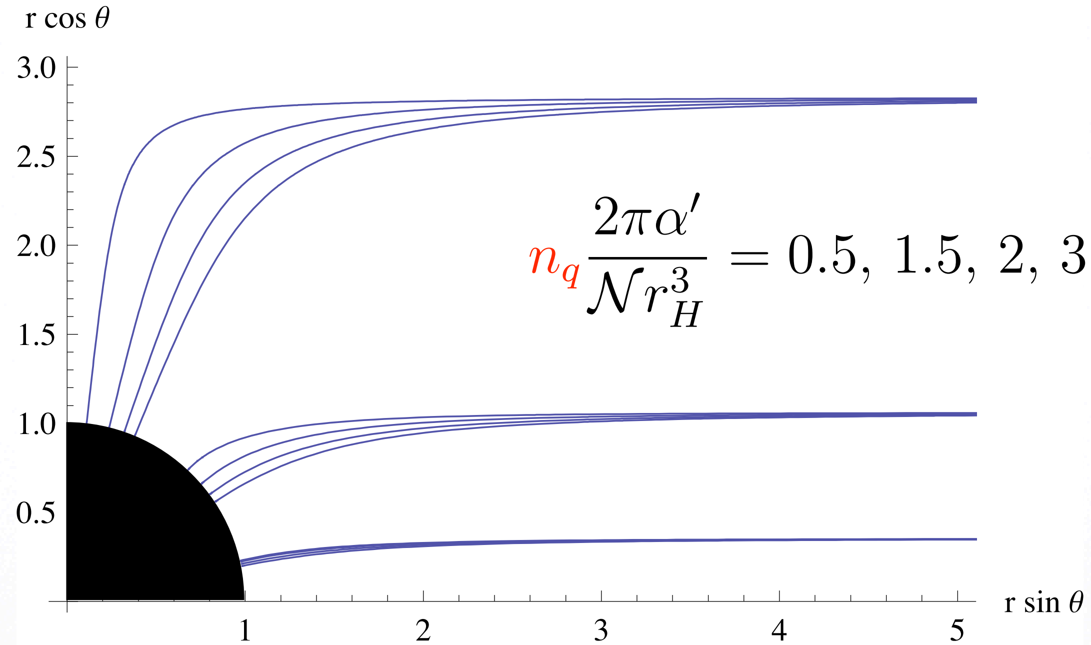


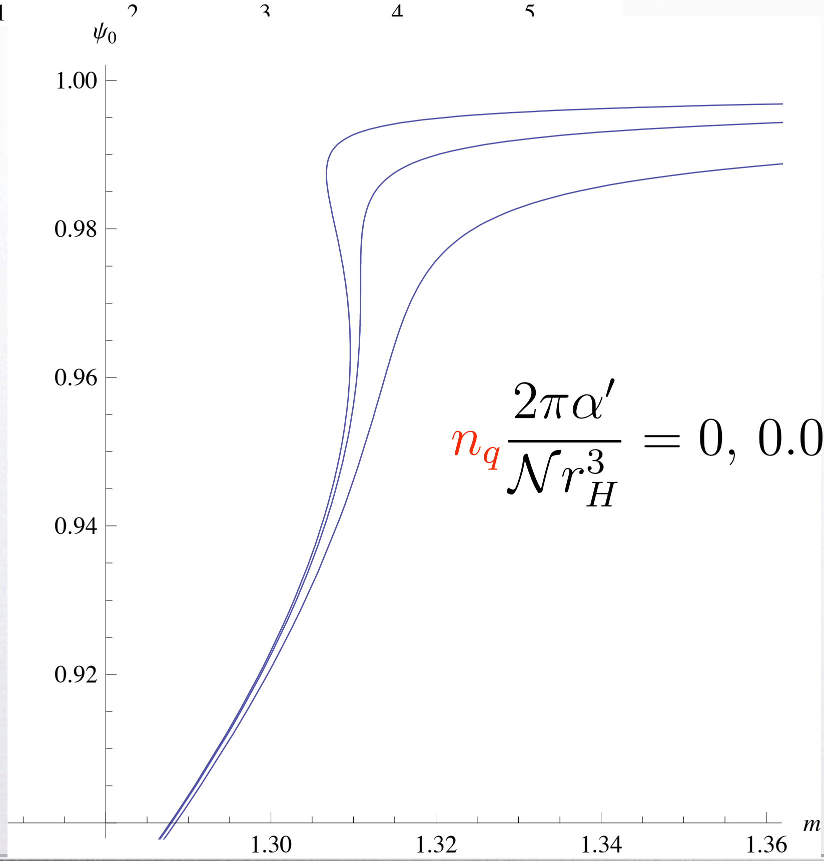
This is interpreted as a condensate of F1 strings.

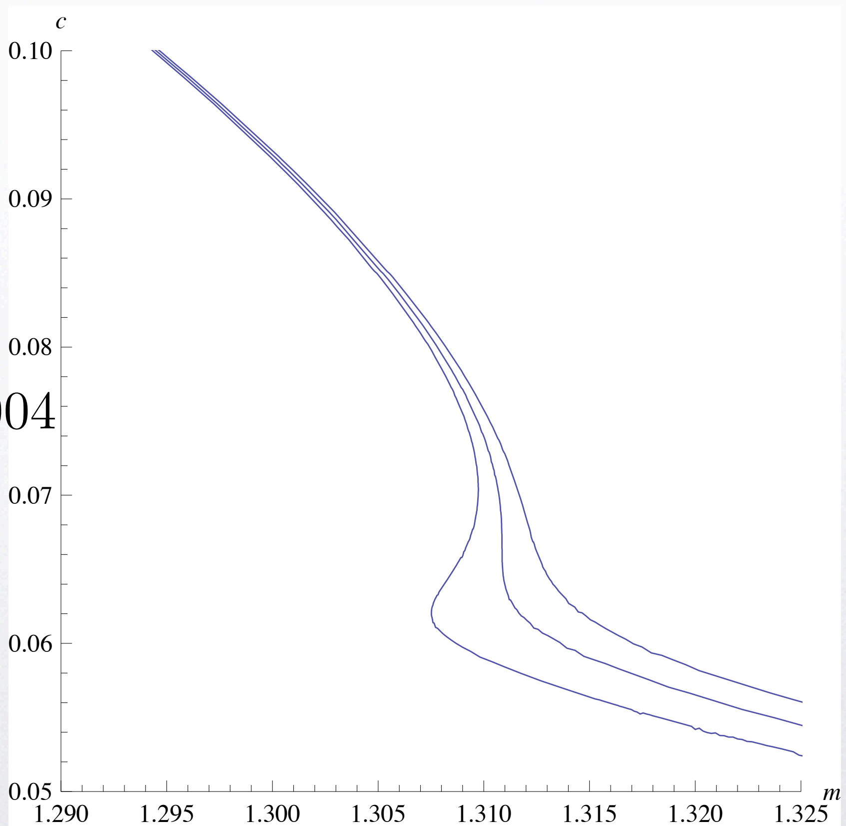
S. Kobayashi, D. Mateos, S. Matsuura, R. Myers & R.Thomson (2007)

Holographic Dp/Dq flavor system: background









Holographic Dp/Dq flavor system: fluctuations

To compute correlators we need the perturbations of the world-volume fields

$$\psi(r,x) \to \psi(r) + \epsilon\, e^{-i(\omega t - qz)}\Psi(r) \quad ; \qquad A_\mu(r,x) \to A_\mu(r) + \epsilon e^{-i(\omega t - qz)}\mathcal{A}_\mu(r)$$

the expansion of the DBI lagrangian to second order gives

$$\mathcal{S}_2 = \frac{\mathcal{N}}{2}\int dx^{q+2}\,\sqrt{-\gamma}\,e^{-\phi}\left(\frac{1}{2}\gamma^{ab}\gamma^{cd} - \gamma^{ad}\gamma^{cb}\right)F_{ab}(\mathcal{A})F_{cd}(\mathcal{A})$$

$$\gamma_{ab} = \tilde{g}_{ab} + (2\pi\alpha')F_{ab}$$

gives the equations of motion

$$E_\perp'' + \partial_r \log\left[e^{-\phi}\sqrt{-\gamma}\gamma^{ii}\gamma^{rr}\right]E_\perp' - \frac{\omega^2\gamma^{00} + q^2\gamma^{ii}}{\gamma^{rr}}E_\perp = 0\,. \qquad E_\perp = \omega\mathcal{A}_\perp$$

$$\begin{aligned} E_{||}'' + \mathfrak{A}_1 E_{||}' + \mathfrak{B}_1 E_{||} + \mathfrak{C}_1\Psi'' + \mathfrak{D}_1\Psi' + \mathfrak{E}_1\Psi &= 0\,, \\ E_{||}'' + \mathfrak{A}_2 E_{||}' + \mathfrak{B}_2 E_{||} + \mathfrak{C}_2\Psi'' + \mathfrak{D}_2\Psi' + \mathfrak{E}_2\Psi &= 0 \end{aligned} \qquad E_{||} = q\mathcal{A}_0 + \omega\mathcal{A}_{||}$$

Conductivity σ

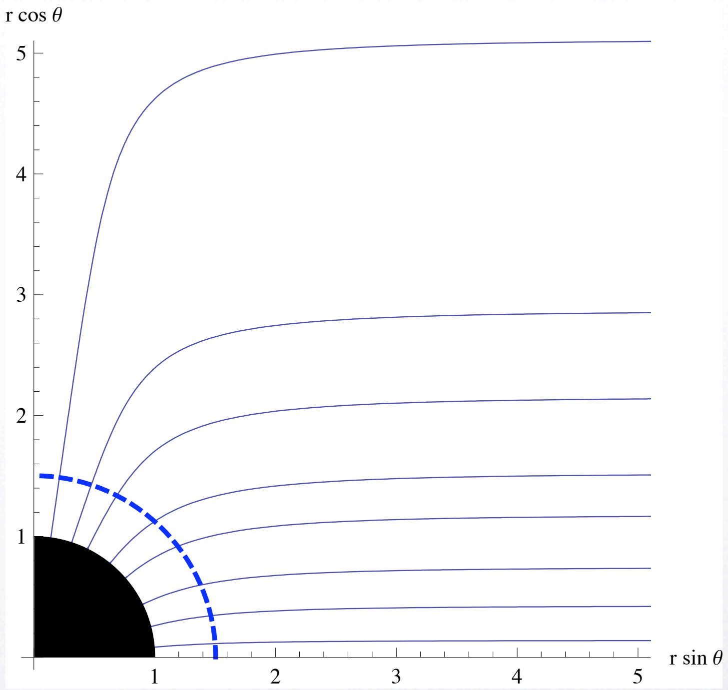
From the transverse correlator we can obtain the conductivity using the Kubo formula

$$\sigma \equiv -\lim_{\omega\to 0} \frac{\mathrm{Im}\,\Pi_\perp(\omega, \mathbf{q}=0)}{\omega}$$

With little more we may capture the whole conductivity tensor $J_i = \sigma_{ij} E_j$

Switch on

$$\begin{aligned} A_0 &= A_t(r) \\ A_1 &= -E_x t + B_z y + e^{-i(\omega t + qz)} \mathcal{A}_x(r) \\ A_2 &= e^{-i(\omega t + qz)} \mathcal{A}_y(r) \\ \vdots &= 0 \end{aligned}$$



E_x introduces a singularity at $r_*(E_x, B_z) > r_H$ so we will set $E_x = 0$

B_z forces to consider the combinations $E_\pm = \omega(\mathcal{A}_x \pm i\mathcal{A}_y)$

$$E_\perp^{\pm\prime\prime} + \log'\left[e^{-\phi}\sqrt{-\gamma}\gamma^{xx}\gamma^{rr}\right] E_\perp^{\pm\prime} - \left(\frac{\omega^2\gamma^{00} + q^2\gamma^{zz}}{\gamma^{rr}} \pm \omega \log'\left[e^{-\phi}\sqrt{-\gamma}\gamma^{xy}\gamma^{0r}\right] \frac{\gamma^{xy}\gamma^{0r}}{\gamma^{xx}\gamma^{rr}}\right) E_\perp^{\pm} = 0$$

Conductivity σ

Solve in the limit $\mathfrak{w} \ll 1 \;;\; \mathfrak{q} = 0$

$$E_\pm = f(r)^{-i\mathfrak{w}/2}\left(E^{(0)}_{\pm,reg} + \mathfrak{w}E^{(1)}_{\pm,reg} + \ldots\right) = E^{(0)}_{\pm,reg} + \mathfrak{w}E^{(1)}_{\pm,reg} - i\frac{\mathfrak{w}}{2}E^{(0)}_{\pm,reg}\log f(r) + \ldots$$

Undo the combination

$$E_\pm = E_x \pm iE_y \qquad E_{x,y}(r) = E^\infty_{x,y} + \int_{r_H}^{r} \frac{C_{x,y} \pm i\omega e^{-\phi}\sqrt{-\gamma}\gamma^{xy}\gamma^{0r}\, E^\infty_{y,x}}{e^{-\phi}\sqrt{-\gamma}\gamma^{xx}\gamma^{rr}} + \tilde{C}_{x,y}\ldots$$

Fix the constants

$$C_{x,y} = -i\omega e^{-\phi}\sqrt{-\gamma}\left(\sqrt{-\gamma^{00}\gamma^{rr}}\gamma^{xx}E^\infty_{x,y} \pm \gamma^{xy}\gamma^{0r}E^\infty_{y,x}\right)\Big|_{r=r_H}$$

Plug into the boundary action

$$S_{Boundary} = -\frac{\mathcal{N}}{2}\int_{r\to r_\infty} d^{d+1}x\frac{e^{-\phi}\sqrt{-\gamma}\gamma^{rr}}{\omega^2}(E_x, E_y)\begin{pmatrix}\gamma^{xx} & \gamma^{xy}\\ -\gamma^{xy} & \gamma^{xx}\end{pmatrix}\begin{pmatrix}E'_x\\ E'_y\end{pmatrix}$$

$$= \int d^{d+1}x(E^\infty_x, E^\infty_y)\begin{pmatrix}-2\Pi_{xx} & -2\Pi_{xy}\\ -2\Pi_{yx} & -2\Pi_{yy}\end{pmatrix}\begin{pmatrix}E^\infty_x\\ E^\infty_y\end{pmatrix}$$

Conductivity σ

$$
\begin{aligned}
\Pi_{xx} &= -i\omega\mathcal{N}\left(\left(e^{-\phi}\sqrt{\gamma\gamma^{00}\gamma^{rr}}\gamma^{xx}\right)_{r_H} + \frac{\gamma^{xy}}{\gamma^{xx}}\left(\left(e^{-\phi}\sqrt{-\gamma}\gamma^{xy}\gamma^{0r}\right)_{r_H} - e^{-\phi}\sqrt{-\gamma}\gamma^{xy}\gamma^{0r}\right)\right)\Big|_{r\to r_\infty} \\
\Pi_{yy} &= \Pi_{xx} \\
\Pi_{xy} &= -i\omega\mathcal{N}\left(\left(\left(e^{-\phi}\sqrt{-\gamma}\gamma^{xy}\gamma^{0r}\right)_{r_H} - e^{-\phi}\sqrt{-\gamma}\gamma^{xy}\gamma^{0r}\right) + \frac{\gamma^{xy}}{\gamma^{xx}}\left(e^{-\phi}\sqrt{\gamma\gamma^{00}\gamma^{rr}}\gamma^{xx}\right)_{r_H}\right)\Big|_{r\to r_\infty} \\
\Pi_{yx} &= -\Pi_{xy}
\end{aligned}
$$

$$
\lim_{r\to r_\infty}\gamma^{xy}/\gamma^{xx} = \lim_{r\to r_\infty}\sqrt{-\gamma}\gamma^{xy}\gamma^{0r} = 0
$$

$$
\sigma_{ij} = \lim_{\omega\to 0}\frac{-\mathrm{Im}\Pi_{ij}}{\omega} \quad\Longrightarrow\quad \boxed{\begin{aligned}\sigma_{xx} &= \mathcal{N}e^{-\phi}\sqrt{\gamma\gamma^{00}\gamma^{rr}}\gamma^{xx}\Big|_{r\to r_H} \\ \sigma_{xy} &= \mathcal{N}e^{-\phi}\sqrt{-\gamma}\,\gamma^{xy}\gamma^{0r}\Big|_{r\to r_H}\end{aligned}}
$$

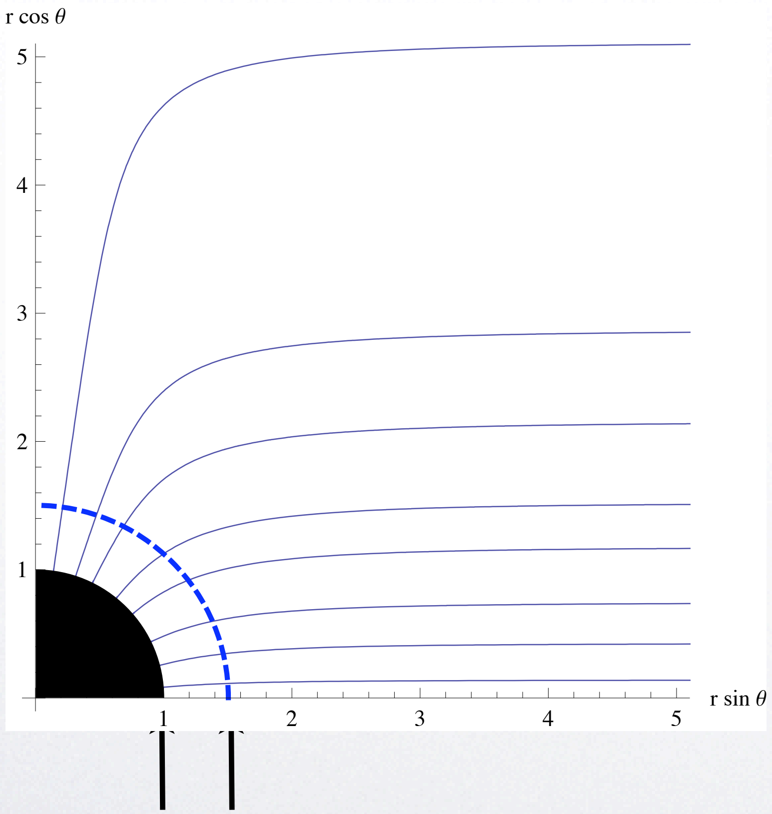
J.M. , J.Tarrío & J.P. Shock (2008)

Conductivity σ

Speculative: how to introduce finite E_x field? .The singular shell appears at position $r_*(E_x, B_z)$ defined by

$$\det(-\gamma_{ij})\Big|_{r=r_*(E_x,B_z)} = 0 \quad ; \quad i,j = (t,x,y)$$

Hence it looks like a horizon of the effective metric. Naively we would just replace $r_H \to r_*(E_x, B_z)$



r_H $r_*(E_x, B_z)$

$$\boxed{\begin{aligned} \sigma_{xx} &= \mathcal{N} e^{-\phi} \sqrt{\gamma\gamma^{00}\gamma^{rr}}\,\gamma^{xx}\Big|_{r\to r_*(E_x,B_z)} \\ \sigma_{xy} &= \mathcal{N} e^{-\phi} \sqrt{-\gamma}\,\gamma^{xy}\gamma^{0r}\Big|_{r\to r_*(E_x,B_z)} \end{aligned}}$$

Conductivity σ from Ohm's law

A. Karch & A. O'Bannon (2007)

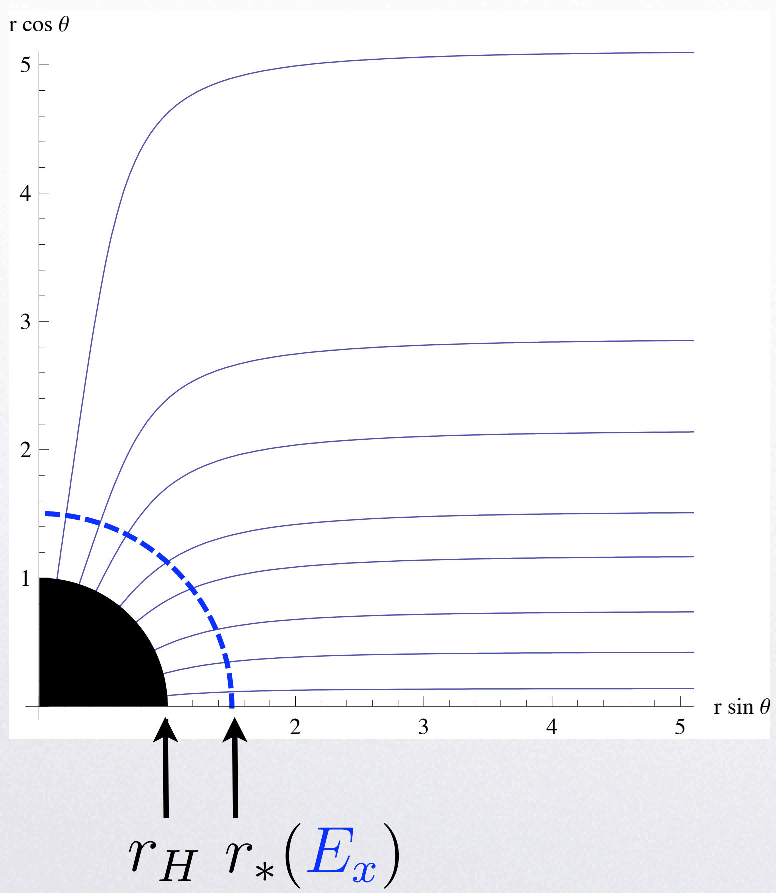
Physical picture:

The DC conductivity measures the response to an external force (Ohm's law)
switch on a macroscopic E_x and B_z and observe how a current flows

$$\begin{cases} J_x = \sigma_{xx} E_x \\ J_y = \sigma_{yx} E_x \end{cases}$$

Holographic picture:

$$A_t(r) = \mu + \frac{n_q}{r} + \dots \qquad A_x = -E_x\, t + a_x(r) \qquad A_y = B_z\, x + a_y(r)$$



Eqs. of motion become imaginary at $r_H \leq r \leq r_*(E_x)$
unless $a_x(r)$ and $a_y(r)$ are turned on

Solving for them, one finds an expansion near $r_\infty = \infty$

$$a_x(r) = \frac{J_x(n_q, E_x, B_z)}{r} + \dots$$

$$a_y(r) = \frac{J_y(n_q, E_x, B_z)}{r} + \dots$$

$$\sigma_{ij} = J_i(n_q, E_x, B_z)/E_j$$

Conductivity σ from Ohm's law

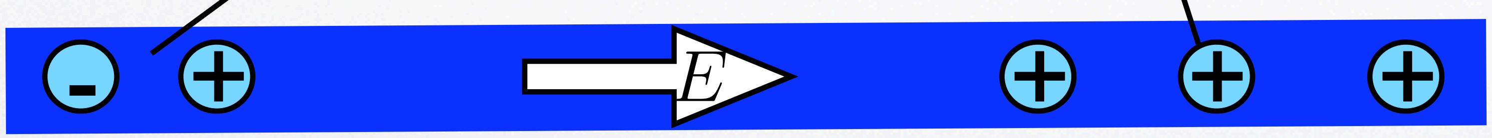
A. O'Bannon (2007)

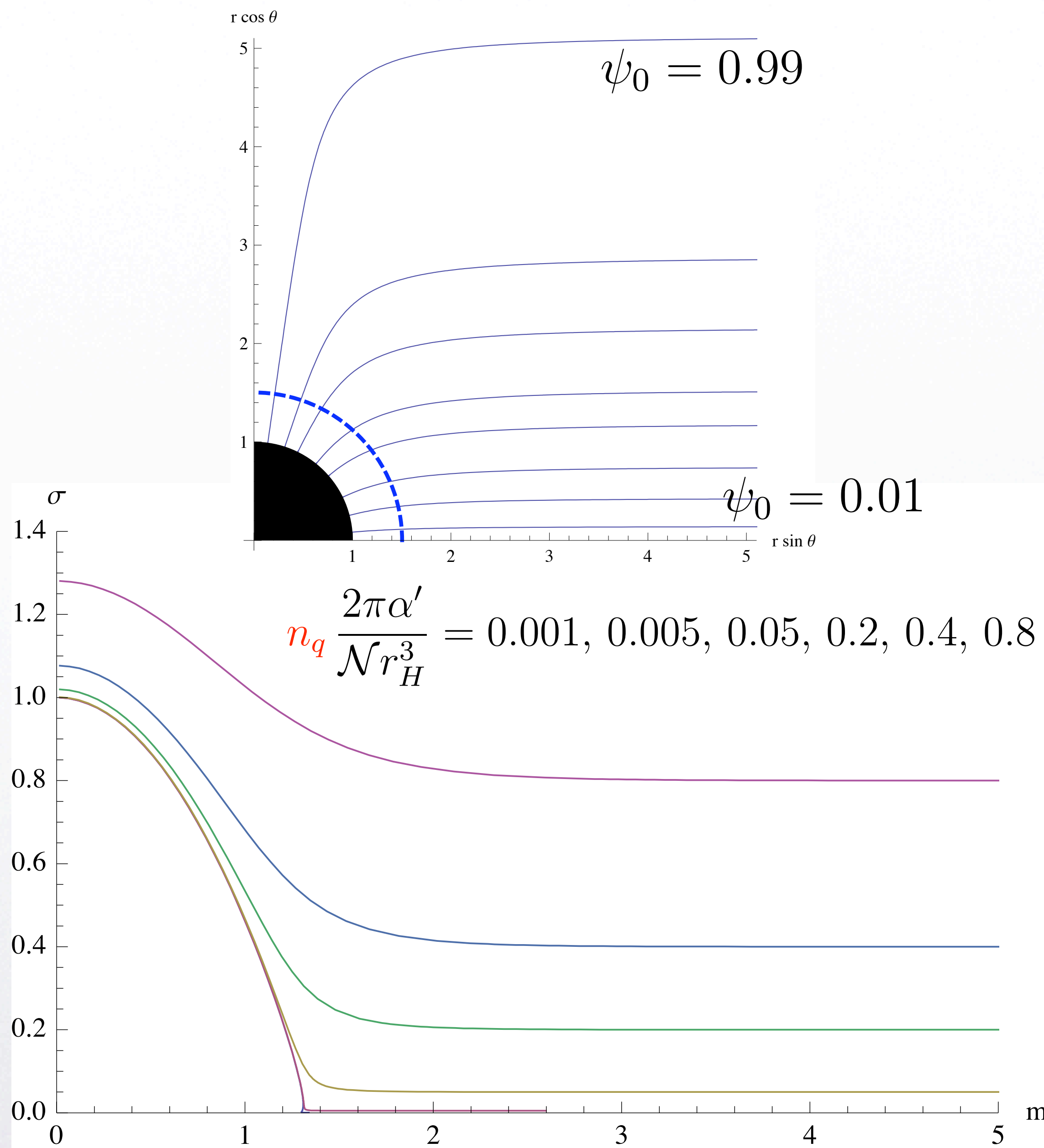
$$\sigma_{xx} = \frac{\tilde{g}_{xx}}{\tilde{g}_{xx}^2 + (2\pi\alpha')^2 {B_z}^2} \sqrt{\mathcal{N}^2 (2\pi\alpha')^4 \left(\tilde{g}_{xx}^2 + (2\pi\alpha') {B_z}^2 \right) e^{-2\phi} \tilde{g}_{xx}^{d-2} \tilde{g}_{\theta\theta}^{n} + (2\pi\alpha')^2 \tilde{g}_{xx}^{-2} {n_q}^2}\ \Bigg|_{r=r_*(E_x, B_z)}$$

$$= \mathcal{N} e^{-\phi} \sqrt{\gamma \gamma^{00} \gamma^{rr}}\, \gamma^{xx} \Big|_{r \to r_*(E_x, B_z)}$$

$B_z = 0$

$$\overset{D3/D7}{\hookrightarrow} \sqrt{\mathcal{N}^2 \sqrt{{E_x}^2 + 1}\,(1 - \psi(r_*)^2)^3 + \frac{{n_q}^2}{{E_x}^2 + 1}}$$





Conductivity σ from Ohm's law

$$\sigma_{xy} = \left.\frac{(2\pi\alpha')^2 n_q B_z}{g_{xx}^2(r) + (2\pi\alpha')^2 {B_z}^2}\right|_{r=r_*(E_x,B_z)}$$

$$= \mathcal{N} e^{-\phi}\sqrt{-\gamma}\,\gamma^{xy}\gamma^{0r}\Big|_{r\to r_*(E_x,B_z)}$$

In the limit $T \to 0 \quad \Rightarrow \quad r_* = r_H(1 + {E_x}^2 + \ldots) \to 0 \quad \Rightarrow \quad g_{xx}(r_*) \to 0$

the relativistic expression is recovered

$$\sigma_{xx} = \sigma_{yy} \to 0 \quad , \quad \sigma_{xy} \to \frac{n_q}{B_z}$$

Charge diffusion: D

In this case we are interested in the hydrodynamic pole of the longitudinal part of the propagator. Hence we study longitudinal fluctuations.

Let us first set $\underline{n_q = 0}$

$$E''_{||} + \partial_r \log\left[\frac{e^{-\phi}\sqrt{-\gamma}\gamma^{ii}\gamma^{rr}}{\omega^2 + q^2\frac{\gamma^{ii}}{\gamma^{00}}}\right] E'_{||} - \frac{\omega^2\gamma^{00} + q^2\gamma^{ii}}{\gamma^{rr}} E_{||} = 0\,.$$

this can be integrated in the hydro limit

$$\lambda \sim \mathfrak{w}, \mathfrak{q}^2$$

$$E_{||}(r) = f(r)^{-i\frac{\mathfrak{w}}{2}}(E^{(0)}_{||,reg} + \lambda E^{(1)}_{||,reg} + \ldots)$$

$$= 1 + i\frac{\mathfrak{q}^2}{\mathfrak{w}}\, e^{-\phi}\sqrt{-\gamma}\gamma^{rr}\gamma^{00} f'\Big|_{r_H} \int_{r_H}^{r} \frac{\mathrm{d}r}{e^{-\phi}\sqrt{-\gamma}\gamma^{00}\gamma^{rr}} + \mathcal{O}(\mathfrak{w}, \mathfrak{q}^2)$$

And we obtain the dispersion relation from the pole in $\Pi_{||}$

$$\Pi_{||} \sim \frac{E'_{||}(r_\infty)}{E_{||}(r_\infty)} \implies E_{||}(r_\infty) = 0 \implies \omega = -iD_0 q^2$$

$$\boxed{D_0 = e^{-\phi}\sqrt{\gamma\gamma^{00}\gamma^{rr}}\gamma^{ii}\Big|_{r_H} \int_{r_H}^{r_\infty} \frac{dr}{e^{-\phi}\sqrt{-\gamma}\gamma^{00}\gamma^{rr}}}$$

Charge diffusion: D

Now we set $\underline{n_q \neq 0}$ The equations of motion mix up $E_{||}$ with the scalar fluctuations Ψ

$$\begin{aligned} E_{||}'' + \mathfrak{A}_1 E_{||}' + \mathfrak{B}_1 E_{||} + \mathfrak{C}_1 \Psi'' + \mathfrak{D}_1 \Psi' + \mathfrak{E}_1 \Psi &= 0\,, \\ E_{||}'' + \mathfrak{A}_2 E_{||}' + \mathfrak{B}_2 E_{||} + \mathfrak{C}_2 \Psi'' + \mathfrak{D}_2 \Psi' + \mathfrak{E}_2 \Psi &= 0 \end{aligned}$$

Still in the hydro limit we can show that

$$E_{||}(r) = 1 + n_q \frac{2\pi\alpha'}{\mathcal{N}} \int_{r_H}^{r} \frac{\Delta \Psi'_{(0)} + \Xi \Psi_{(0)}}{\sqrt{-\gamma}\gamma^{00}\gamma^{rr}} dr + i \frac{\mathfrak{q}^2}{\mathfrak{w}} \, e^{-\phi}\sqrt{-\gamma}\gamma^{rr}\gamma^{00} f'\Big|_{r_H} \int_{r_H}^{r} \frac{\mathrm{d}r}{e^{-\phi}\sqrt{-\gamma}\gamma^{00}\gamma^{rr}} + \mathcal{O}(\mathfrak{w}, \mathfrak{q}^2)$$

with $\Delta = \Delta(\gamma_{ab}, \psi)$; $\Xi = \Xi(\gamma_{ab}, \psi)$ From here, we obtain the pole at $\omega = -iDq^2$

$$D = \frac{D_0}{1 + n_q \dfrac{2\pi\alpha'}{\mathcal{N}} \displaystyle\int_{r_H}^{r_\infty} \frac{\Delta \Psi'_{(0)} + \Xi \Psi_{(0)}}{\sqrt{-\gamma}\gamma^{00}\gamma^{rr}}}$$

At zero quark mass $\psi = 0 \quad \longrightarrow \quad \Delta(\gamma_{ab}, 0) = \Xi(\gamma_{ab}, 0) = 0$ hence $D = D_0$

At generic quark mass we don't have an expression in terms of background quantities (γ_{ab}, ψ)

if Einstein relation holds $D = \sigma/\chi$ then we have a bonus

Susceptibility χ

The holographic dual expression for the chemical potential is

$$\mu = A_0(r_\infty) - A_0(r_H) = \int_{r_H}^{r_\infty} A_0'(r)dr$$

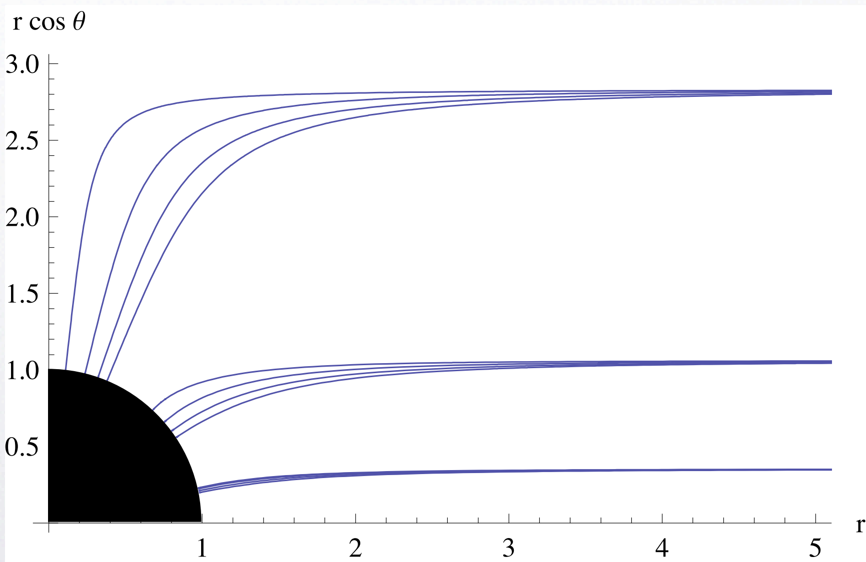
With this we can express the thermodynamical susceptibility

$$\chi = \left.\frac{\partial n_q}{\partial \mu}\right|_T = \left(\frac{\partial \mu}{\partial n_q}\right)_T^{-1} = \left(\int_{r_H}^{r_B} \frac{dA_0'(r)}{dn_q} dr\right)^{-1}$$

Hence we need to invert $n_q(A_0')$ and solve for A_0'

$$A_0'(r) = \frac{n_q}{\mathcal{N}} \sqrt{\frac{-\gamma_{00}\gamma_{rr}}{e^{-2\phi}\gamma_{ii}^p\gamma_{\theta\theta}^n + (2\pi\alpha')^2 \frac{n_q{}^2}{\mathcal{N}^2}}}\,.$$

Notice that there is also an implicit dependence through $\psi(n_q)$



$$\boxed{\chi = \mathcal{N}\left(\int_{r_H}^{r_\infty} \frac{1}{e^{-\phi}\sqrt{-\gamma}\gamma^{00}\gamma^{rr}}\left[1 + n_q\left(\Delta\frac{\partial \psi'}{\partial n_q} + \Xi\frac{\partial \psi}{\partial n_q}\right)\right]\right)^{-1}}.$$

Einstein relation $D = \frac{\sigma}{\chi}$

$$\boxed{D = \underbrace{\left(e^{-\phi}\sqrt{\gamma\gamma^{00}\gamma^{rr}}\gamma^{ii}\Big|_{r_H} \int_{r_H}^{r_\infty} \frac{dr}{e^{-\phi}\sqrt{-\gamma\gamma^{00}\gamma^{rr}}} \right)}_{D_0} \frac{1}{1 + n_q \frac{2\pi\alpha'}{\mathcal{N}} \int_{r_H}^{r_\infty} \frac{(\Delta\tilde{\Psi}'_{(0)} + \Xi\tilde{\Psi}_{(0)})}{\sqrt{-\gamma\gamma^{00}\gamma^{rr}}}}}$$

$$\boxed{\sigma = \mathcal{N} e^{-\phi}\sqrt{\gamma\gamma^{00}\gamma^{rr}}\gamma^{ii}\Big|_{r_H}}$$

$$\boxed{\chi = \mathcal{N} \left(\int_{r_H}^{r_\infty} \frac{1}{e^{-\phi}\sqrt{-\gamma\gamma^{00}\gamma^{rr}}} \left[1 + n_q \left(\Delta \frac{\partial\psi'}{\partial n_q} + \Xi \frac{\partial\psi}{\partial n_q} \right) \right] \right)^{-1}}.$$

At vanishing baryon number $n_q = 0$ and/or mass $m = 0 \Rightarrow \Delta = \Xi = 0$

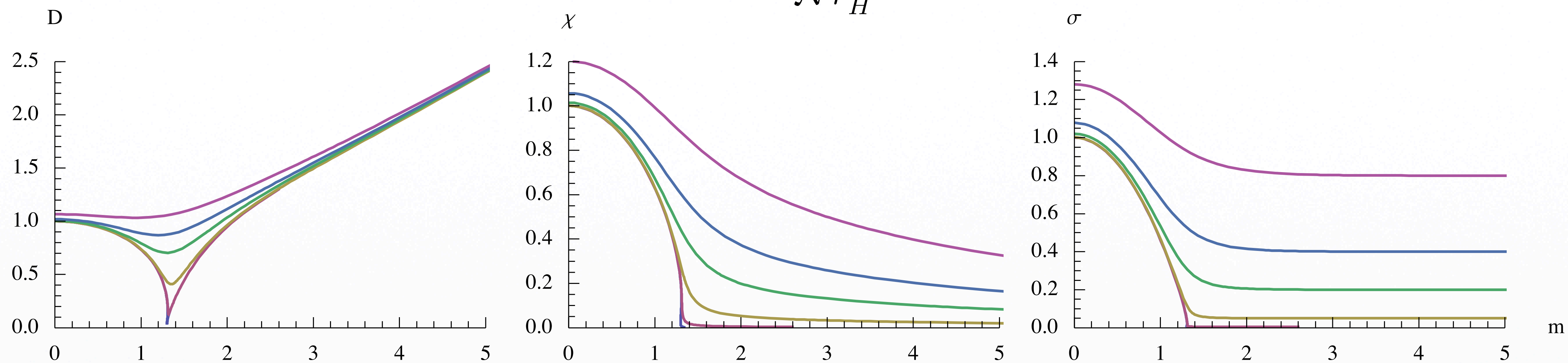
$$D_0 = \frac{\sigma}{\chi}$$

holds analytically

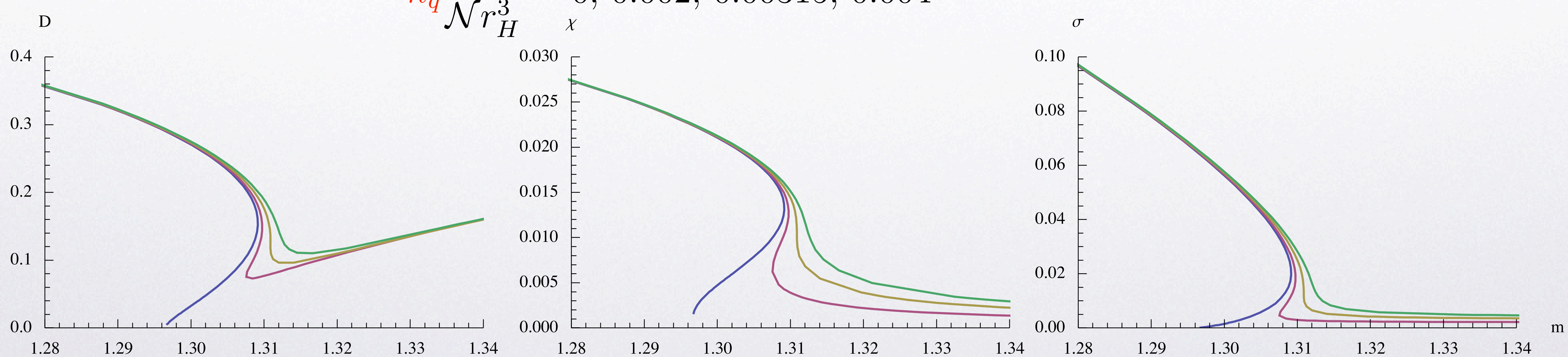
At non-vanishing baryon number $n_q \neq 0$ and mass $m \neq 0$ we need to resort to numerics

For the particular case of the D3/D7 with

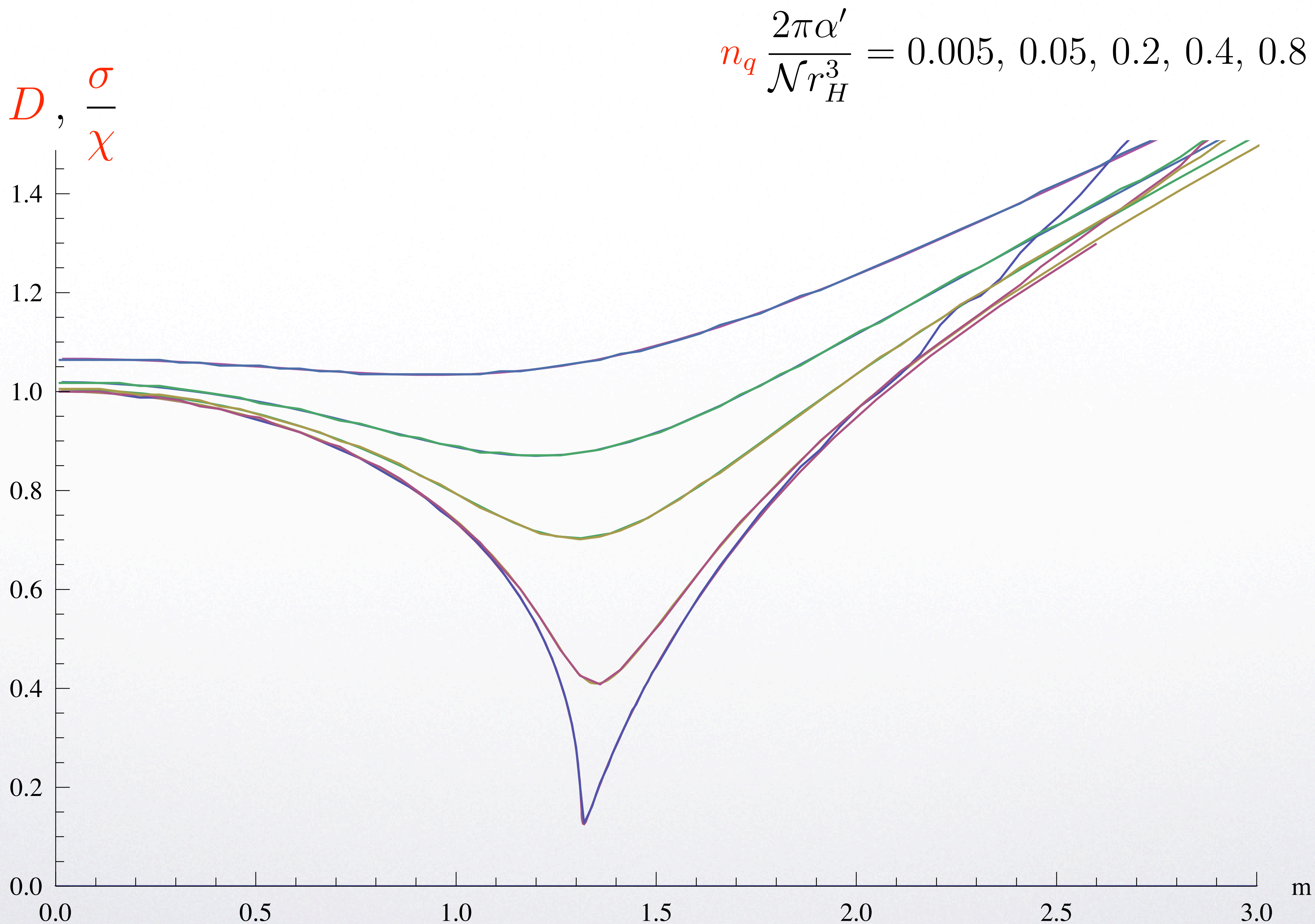
$$n_q \frac{2\pi\alpha'}{\mathcal{N} r_H^3} = 0.001,\ 0.005,\ 0.05,\ 0.2,\ 0.4,\ 0.8$$



$$n_q \frac{2\pi\alpha'}{\mathcal{N} r_H^3} = 0,\ 0.002,\ 0.00315,\ 0.004$$



Einstein relation $D = \frac{\sigma}{\chi}$



SUMMARY AND OPEN ISSUES

- We provided general expressions for σ in terms of background quantities $(\gamma_{ab}(r), \psi(r))$ in presence of finite (E_x, B_z) and found agreement with the macroscopic analysis.

- We have checked Einstein relations for massless flavors and also for massive flavors in the D3/D7 system. This permits to write a general formula for the diffusion constant on Dp/Dq flavor systems

$$D = e^{-\phi}\sqrt{\gamma\gamma^{00}\gamma^{rr}}\gamma^{ii}\Big|_{r\to r_H} \times \int_{r_H}^{r_\infty} \frac{1}{e^{-\phi}\sqrt{-\gamma}\gamma^{00}\gamma^{rr}}\left[1+n_q\left(\Delta\frac{\partial\psi'}{\partial n_q}+\Xi\frac{\partial\psi}{\partial n_q}\right)\right]$$

- Other generalized Einstein relations can be also checked

$$D_T = \sigma_T/C_p \qquad\qquad D_p = \eta/\chi_p$$

- For finite E we must make fluctuation analysis in presence of a singular shell. Interesting even at zero temperature.