# Cancellations of ultraviolet divergences in supergravity

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IPhT - CEA/Saclay

June 18, 2008 @ "Windows on Quantum Gravity"

based on

- arXiv:0802.0868 and arXiv:0805.3682 with N.E.J.Bjerrum-Bohr
- hep-th/0611273 and hep-th/0610299 with M.B. Green, J.G.Russo

# Outline

1. **UV behaviour of gravity amplitudes**
   - Pure gravity amplitudes
   - $\mathcal{N}=8$ amplitudes
2. **On-shell supersymmetry**
3. **The no triangle property of $\mathcal{N}=8$ supergravity**
4. **Conclusion & Outlook**

Part I

# UV behaviour of gravity amplitudes

# UV behaviour of gravity amplitudes

Gravity describes the interactions of a massless spin 2 particle with a dimensionfull coupling constant

$$[\kappa^2_{(D)}] = (length)^{D-2}$$

A $L$-loop 4-point *pure* gravity amplitude in dimensions $D$ has the mass

$$[\mathfrak{M}_L^{(4)}] = \text{mass}^{(D-2)L+2}$$

# UV behaviour of pure gravity amplitudes

- At one-loop $L = 1$ the amplitude is diverging with for counter-term

$$\mathfrak{M}_1^{(4)} \sim \frac{1}{\epsilon}\,[\alpha R^2_{mnpq} + \beta R^2_{mn} + \gamma R^2], \qquad D = 4 - 2\epsilon$$

  - In 4d $R^2_{mnpq} \sim 4R^2_{mn} - R^2$, and for *pure* gravity $R_{mn} = 0 = R$ so the divergence is zero *on-shell* 't Hooft/Veltman

- At two-loop $L = 2$ Sagnotti/Goroff; van de Ven

$$\mathfrak{M}_2^{(4)} \sim \frac{1}{\epsilon}\,(\kappa^2_{(4)} R_{mnpq})^3$$

- At $L$-loop order a new counter-term arises

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# UV behaviour of supergravity amplitudes

## $\mathcal{N} = 8$ supergravity

- The low-energy limit of $L$-loop amplitudes

$$[\mathfrak{M}_L^{(D)}] = \mathrm{mass}^{(D-2)L-6-2\beta_L}\, D^{2\beta_L} R^4$$

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$$D \geq D_c = 2 + \frac{6 + 2\beta_L}{L}$$

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- String duality arguments and explicit computations indicate that
  Green, Russo, Vanhove

$$\beta_L = L \text{ for } L \geq 2$$

# UV behaviour of multiloop amplitudes

When $\beta_L = L$ the amplitude behaves has Green,Russo,Vanhove

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$$[\mathfrak{M}_L^{(D)}] = \text{mass}^{(D-4)L-6}\, D^{2L} R^4$$

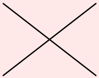
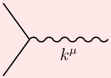
- Non-renormalisation theorems of $D^{2g} R^4$ after genus-$g$ string theory
  - $R^4$ (1-loop), $D^4 R^4$ (2-loop), $D^6 R^4$ (3-loop), Green et al.
  - confirmed by explicit computation to $g \leq 5$ Berkovits
- If true for all $L$ then the theory is perturbatively finite in 4d

$$D \geq D_c = 4 + \frac{6}{L}$$

  - The UV dependence is *the same as for $\mathcal{N} = 4$ SYM amplitudes*
  - Although it is not possible to decouple $\mathcal{N} = 8$ supergravity from string theory in $D \geq 4$ Green,Schwarz,Ooguri

# Structure of $\mathcal{N} = 8$ gravity amplitudes

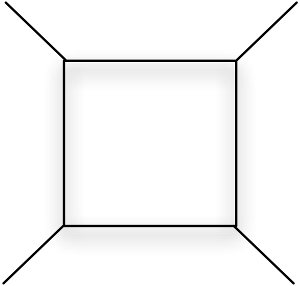
$$[\mathfrak{M}_L^{(4)}] = \mathrm{mass}^{(D-4)L-6}\, D^{2L} R^4 \text{ indicates ``effective'' interactions}$$

 : $\varphi^4$ or 

: $\varphi^2$coupled to electromagnetism

$\mathcal{N} = 8$ Supergravity amplitudes are expected to be expandable on the *same* basis of integral functions as $\mathcal{N} = 4$ SYM

# UV divergences at higher-loop



One loop amplitude is given by $\varphi^3$ scalar box amplitude with $\beta_1 = 0$

$$\mathfrak{M}^{(D)}_{4;1} = R^4\ [I_{\rm box}(s,t) + I_{\rm box}(s,u) + I_{\rm box}(t,u)]$$

Green,Schwarz,Brink

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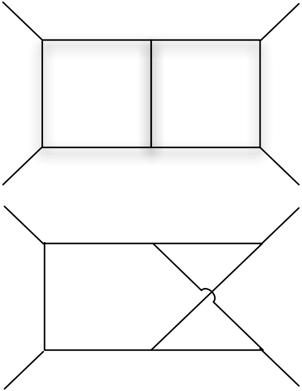


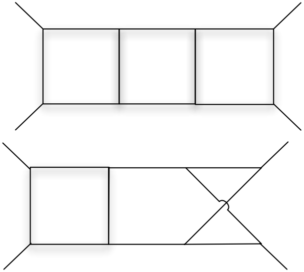
Two-loop amplitude is given by the $\varphi^3$ scalar planar and non-planar doublebox amplitude with $\beta_2 = 2$

$$\begin{aligned}\mathfrak{M}_{4;1}^{(D)} &= D^4 R^4 \Big[ I^{P}_{\text{doublebox}}(s) + I^{NP}_{\text{doublebox}}(s) \\ &+ (t-, u - \text{channels}) \Big]\end{aligned}$$

Bern et al.; D'Hoker, Phong

Berkovits; Berkovits, Mafra

# UV divergences at higher-loop

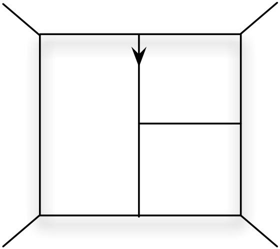


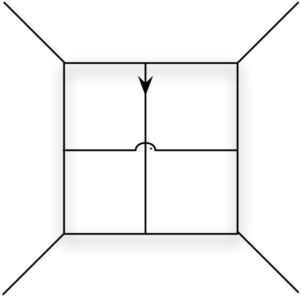
Higher-loop $\varphi^3$ scalar *ladder* contributions behaves as $\beta_L = 2(L-1)$

$$\mathfrak{M}_{4;1}^{(D)} = D^{4L-2} R^4 \left[ I^P_{\text{ladder}}(s) + I^{NP}_{\text{ladder}}(s) + (t-, u - \text{channels}) \right]$$

which is too much converging for being the leading $\mathcal{N} = 8$ UV behaviour

# UV divergences at higher-loop





higher-loop have non-ladder topologies which *individually* do *not* behaves as $\beta_L = L$

$$= D^4 R^4 \int d^D\ell_1 d^D\ell_2 d^D\ell_3 \, \frac{(\ell_1 \cdot k_1)(\ell_2 \cdot k_2}{(\ell_1 - k_1)^2 \cdots}$$

Bern, Dixon,Roiban

Bern, Carrasco, Dixon, Johansson, Kosower, Roiban

Berkovits' gravitational F-terms satisfy the rule $\beta_L = L$

Part II

# On-shell supersymmetry

# On-shell Supersymmetry and counter-terms

(On-shell) Extended Supersymmetry improves UV behaviour

Grisaru, Deser, Stelle, Howe, Lindström, Kallosh, ...

- $\mathcal{N} \geq 1$ susy forbids $R^3$ counterterms: supergravity is finite at $L = 2$
- for $\mathcal{N} \leq 4$ susy a *first possible* counter-terms at $L = 3$ loop order

$$S^{(3)}_{\mathcal{N}\leq 4} = \int d^4x \int d^{4\mathcal{N}}\theta\, E\, (\varphi + \cdots + \theta^{\mathcal{N}} R)^4 = \int d^4x \sqrt{g}\, (\kappa^2_{(4)} R)^4$$

- for $\mathcal{N} \geq 4$ a *first possible* counter-term at $L = \mathcal{N} - 1$ loop order

$$S^{(\mathcal{N}-1)}_{\mathcal{N}\geq 4} = \int d^4x \int d^{4\mathcal{N}}\theta\, E\, (\varphi + \cdots + \theta^4 R)^4 = \int d^4x \sqrt{g}\, (\kappa^2_{(4)})^{\mathcal{N}} D^{2(\mathcal{N}-4)} R^4$$

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The absence of 3-loop divergences in 4d in the four-graviton amplitude found by Bern et al. means that $\mathcal{N} \geq 5$ on-shell supersymmetries are at work

# Gravitational F-terms

Berkovits' pure spinor formalism gives that four-graviton amplitudes are **F-terms** up-to genus six contructed from the superfields

$$\mathfrak{M}_1 \sim \int d^{16}\theta_L d^{16}\theta_R \, \theta_L^{11}\theta_R^{11} \, AW^3 \, I_1 \sim R^4 \, I_1$$
$$\mathfrak{M}_h \sim \int d^{16}\theta_L d^{16}\theta_R \, \theta_L^{12-2h}\theta_R^{12-2h} \, W^4 \, I_L + \cdots \sim D^{2h} R^4 \, I_L + \cdots$$

- constructed from the spin 1 RR-field chiral superfields

$$W_{\alpha\beta,a_1a_2} = F_{\alpha\beta,a_1a_2} + \cdots + \theta^{\gamma}_{a_1}\theta^{\delta}_{a_2} \, R_{\alpha\gamma\beta\delta} + \cdots$$
$$A_{\alpha\beta,a_1a_2} = \theta^{\gamma}_{a_1}\theta^{\delta}_{a_2} g_{\alpha\gamma\,\beta\delta} + \cdots$$

# On-shell Supersymmetry and counter-terms

After some order in derivative supersymmetry runs out of steam and no more powers of momenta is factorized out of the amplitude leading to a formula

$$D \geq D_c = 2 + \frac{\beta_L}{L}; \quad 6 \leq \beta \leq 18$$

Which gives a possible first divergence

$$3 \leq L \leq 9, \qquad D = 4$$

- The nine-loop counter-term

$$S_9^{(4)} \sim \int d^4x \sqrt{g} \int d^{16}\theta_L d^{16}\theta_R \, E \, W^4 \sim \int d^4x \sqrt{g} \, \kappa_{(4)}^{20} D^{12} R^4$$

Part III

# The no triangle property of $\mathcal{N} = 8$ supergravity

# The no triangle property in $\mathcal{N} = 8$

## Basis of scalar integral functions

In $D = 4 - 2\epsilon$ one expands the amplitudes on a basis of scalar integral functions

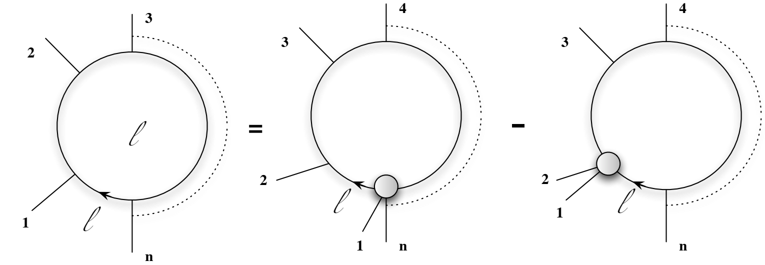
$$\mathfrak{M}_{n;1} = \sum_i b_i\, I_{\Box}^{(i)} + \sum_i t_i\, I_{\triangleright}^{(i)} + \sum_i b_i\, I_{\circ}^{(i)} + c_{\text{rational pieces}}$$

The no triangle hypothesis formulated by Bjerrum-Bohr, Dunbar et al. states the graviton amplitude in $\mathcal{N} = 8$ are only expanded on scalar box integral functions only (the *only box property*).

## Reduction formulas

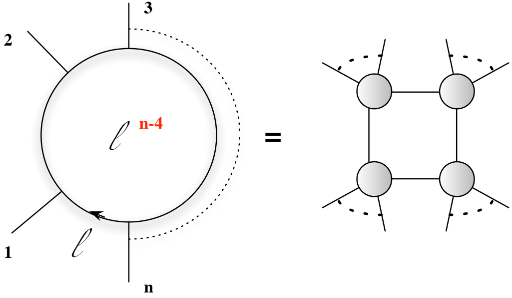
On-shell integral satisfy reduction formulas stating that one can trade powers of loop momentum to massive legs ($k_i^2=0$)

$$\int d^D\ell \, \frac{2(\ell\cdot k_1)}{\ell^2\,(\ell-k_1)^2}\times(\cdots) = \int d^D\ell\,\left(\frac{1}{(\ell-k_1)^2}-\frac{1}{\ell^2}\right)\times(\cdots)$$



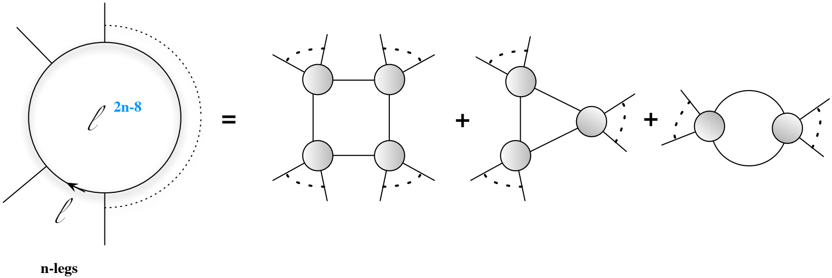
# The no triangle property in $\mathcal{N} = 8$

Since $\mathcal{N} = 4$ super-Yang-Mills amplitudes have $n - 4$ powers of loop momenta they are reducible to boxes only



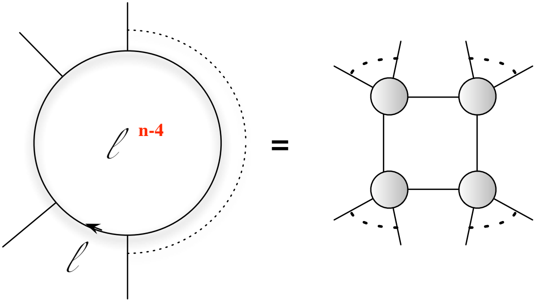
# The no triangle property in $\mathcal{N} = 8$

$\mathcal{N} = 8$ amplitudes $2n - 8$ powers of loop momenta should contains boxes, triangles, bubbles and rational terms



# The no triangle property in $\mathcal{N} = 8$

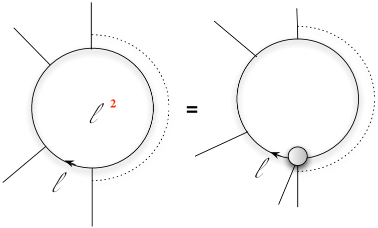
Explicit computations by Bjerrum-Bohr et al., Bern et al. indicated only $n - 4$ like $\mathcal{N} = 4$ SYM and the amplitude is only reducible one scalar box integral functions



# Proofs of the no triangle property in $\mathcal{N} = 8$

- Gravity does not have color factor
  - summation over all the permutations at one-loop
  - Sum over all the planar and nonplanar diagrams at higher loop order
- Gauge invariance $\varepsilon_{\mu\nu} \to \varepsilon_{\mu\nu} + \partial_\mu v_\nu + \partial_\nu v_\mu$

This implies new reduction formulas for unordered integrals




Gauge invariance implies that one can push all the 'triangles' into total derivative which cancel in the total amplitude (no boundary contributions)

Bjerrum-Bohr, Vanhove

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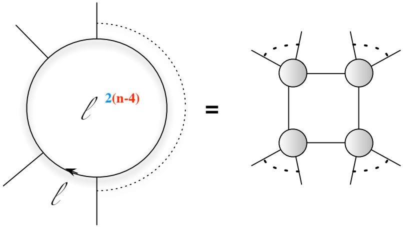
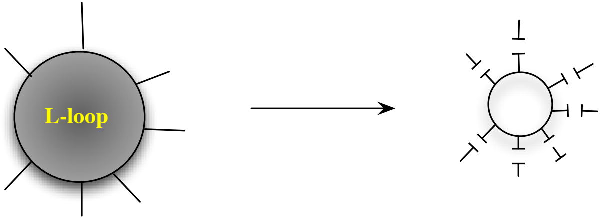
Two powers of loop momenta are cancelled at each step

Bjerrum-Bohr, Vanhove



# Gravity theories with less or not supersymmetry




Since the **extra** cancellations are not due to supersymmetry they occur as well in theories with less or no supersymmetry

- $\mathcal{N}=4$ would have a critical dimension for UV divergence
  - Are reducible to bubbles at one-loop
  - satisfies the rule $\beta_L = L/2$ with a critical dimension $D \geq D_c = 3 + \frac{6}{L}$
- Pure gravity one-loop amplitude would be at most logarithmically diverging like QCD

Bjerrum-Bohr, Vanhove

# Summary & Outlook

We have explained that colorless gauge theory like gravity exhibit important cancellations in on-shell amplitudes. Bjerrum-Bohr,Vanhove

- These cancellation are needed for the $\beta_L = L$ rule to all loop orders
  - Needed to go beyond the six genus limit non-renormalisation in pure spinor string theory
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